Images of the Ammann-Beenker Tiling

Edmund Harriss
Department of Mathematics
Imperial College London
LONDON, SW7 2AZ, United Kingdom
E-mail: edmund.harriss@mathematicians.org.uk

The Ammann-Beenker tiling is the eight-fold sibling of the more famous, five-fold Penrose rhomb tiling. It was discovered independently by R. Ammann [AGS92] and F. Beenker [Bee82]. It shares many properties with the Penrose tiling. In particular it shares two particular constructions. The first by a substitution rule[1], and the second as a slice of a higher dimensional lattice. Also it shares the important property with the Penrose that it is a strikingly beautiful tiling.

R. Ammann's discovery of the tiling was at the same time as Penrose's. In fact he sent in several examples in response to Martin Gardener's Scientific America article that announced the Penrose tiling [Gar77]. Though it was only later that something was published more formally, as Ammann was very detached from the mathematical community[Sen04]. Like Penrose (and nearly all the early aperiodic tilings) he used a substitution rule to construct the tiling.

A substitution rule has two phases. First the tiles are expanded. Then the new larger tiles are replaced by copies of the original tiles. By repeating this process larger and larger patches of tiles can be generated and, at the infinite limit, tilings of the whole plane. The substitution rule for the Ammann tiling is show in Figure 1. In this case the expansion is multiplication by $1 + \sqrt{2}$.

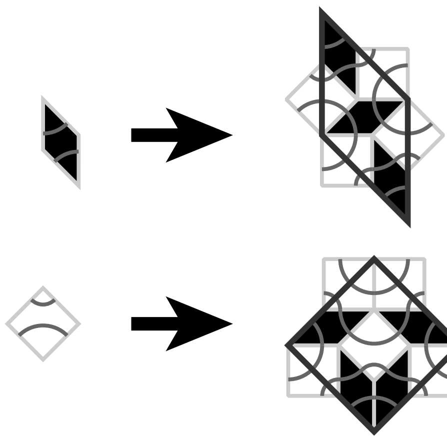

Figure 1: The substitution rule for the Ammann-Beenker Tiling

In 1981, the Dutch mathematician N.G. de Bruijn found an alternative construction for the Penrose tiling [dB81]. His student N. Beenker followed his method for the eight-fold case and found another tiling with substitution rule[2]. This method is related to the way a computer draws a line on the monitor. As the monitor is made up of square pixels it is not possible just to draw the line. Instead one must choose a sequence of pixels that stay close to the line. Looking at the top or bottom of these pixels gives a staircase, with vertical and horizontal lines. We may then project the staircase to the original line giving a tiling with two tiles, corresponding to the projection of the vertical and horizontal lines. In a similar way one may construct a plane out of hypercubes in five (for the Penrose) or four (for the Ammann-Beenker) dimensions, and project one face of this to a

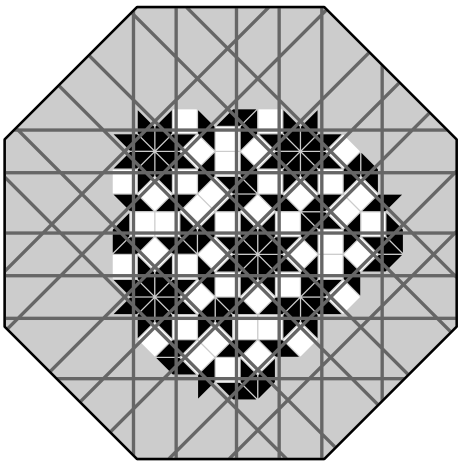

Figure 2: A portion of the Ammann Tiling showing the Ammann bars.

[1] The nomenclature here is a mess with many terms like self-similar tiling, inflate and subdivide and expansion being used.

[2] In fact one can also do the same thing for the 12-fold setting to obtain the Socolar tiling [Soc89], but those three are the only ones that can be obtained by this method with high symmetry. If one uses a less symmetrical setting an infinite number of substitution tilings may be obtained [Har03]

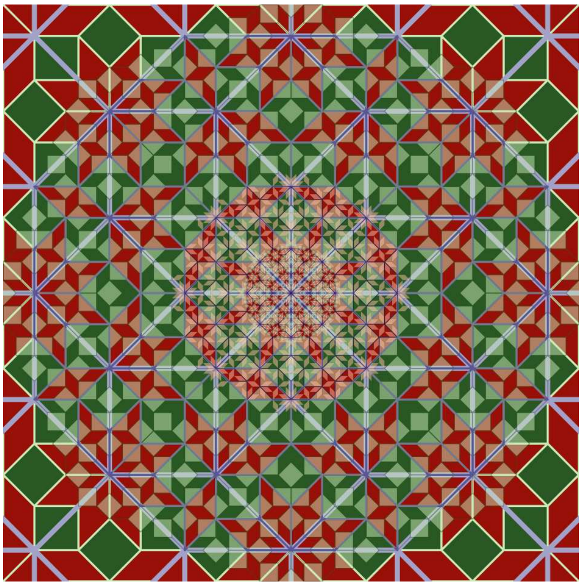

Figure 3: Ammann Scaling

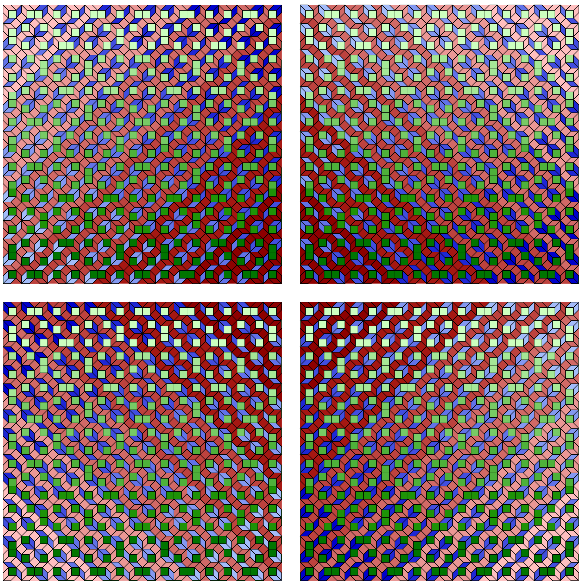

Figure 4: Ammann Squares

plane. If the plane is positioned correctly the tilings produced by this are precisely the Penrose Rhomb, or Ammann-Beenker tilings. As shown in Figure 2, one consequence of this is that the tiles line up into "worms", lying along parallel sets of lines at 45 deg angles, called Ammann bars.

Figures 3 and 4 show artistic (or at least pretty) images that explore the two construction methods of the tiling. Figure 3 overlays several layer of the hierarhcical structure of the tiling, thus illustrating how the substitution rule generates larger patches of tiling. Figure 4 has tiles on the same line with the same colour for three of the four directions (as every tile lies on two lines, after colouring three of the directions, every tile is coloured).

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