Ancient Harmonic Law

Jay Kappraff
New Jersey Institute of Technology
Newark, NJ 07102
kappraff@verizon.net

Abstract

The matrix arithmetic for ancient harmonic theory is presented here for two tuning systems : "Spiral fifths" as presented by Nicomachus, a Syrian Neo-Pythagorean of the second century A.D., and Plato's "Just tuning" as reconstructed by the ethnomusicologist, Ernest McClain, from clues preserved by Nicomachus and Boethius (6th c. AD). These tables lie behind the system of architectural proportions used during the Renaissance, and their basic ratios now pervade modern theories of music. Calculation employs an early form of log table governed by vectors of 2-3-4 in the first, and by 3-4-5 in the second. These systems govern 12-tone theory from the perspective of four primes--2, 3, and 5 generate all ratios under the overview of 7—as disciplined "self-limitation" within a "balance of perfect opposites. "

1. Introduction

Nicomachus was a Syrian mathematician writing about 150 A.D. His work forms one of the best links to what survived from his day about Greek theory of numbers and music [1]. I shall describe how the sequence of integers shown in Table 1, and attributed to Nicomachus, defines musical octaves, fifths, and fourths the only consonances recognized by the Greeks, and lies at the basis of ancient musical scales sometimes attributed to Pythagoras. A second Table, Table 2 inferred by Plato but brought to light by the ethnomusicologist, Ernest McClain [2,3,4], will be shown to be the basis of the Just scale, another ancient musical scale. This table, which I shall refer to as the McClain Table, will also provide a link to the modern theory of music. In his books and papers McClain has made a strong case for music serving as the lingua franca of classical and sacred texts, providing plausible explanations to otherwise difficult to understand passages and providing metaphors to convey ideas and meaning.

2. The Tables

The twelve tones of the chromatic musical scale can be organized around a tone circle (not shown) and numbered from 0 to 11, also called digital roots. Each tone represents a pitch class of tones that differ in relative string length or relative frequency by an octave power of 2. On a logarithmic scale, the tonal intervals of the equal-tempered scale are all equal while they differ when the tones are reckoned in terms of relative string length.

The integers in Table 1 are relative string lengths of tones from the 12 tone, chromatic scale of Western music. All integers are divisible by only primes 2 and 3. The integers are organized so that the octave ratio 2:1 occurs along the rows, the ratio of 3:2, the musical fifth is expressed along the columns, and the musical fourth, 4:3 are along the right leaning diagonals The five successive musical fifths in column 5 constitute a pentatonic scale defined from the middle with fundamental at D, while the seven successive musical fifths in column 7 form a heptatonic scale, with middle tone D as fundamental. If this table is extended to thirteen columns the integers form the tones of the chromatic scale from and g ♯ to a $_b$ with integers extending from 4096 to 531,441.These scales are sometimes attributed to Pythagoras. On an equal tempered scale a $_b$ and g ♯ are

Table 1. Nicomachomachus' table for expansions of the ratio 3:2

(as string lengths, and with modern tone names interpolated)

1	2	4	[A]	8	16	[E]	32	64	[B]
	3	6	[D]	12	24	[A]	48	96	[E]
		9	[G]	18	36	[D]	72	144	[A]
				27	54	[G]	108	216	[D]
					81	[C]	162	324	[G]
(Subsets are viewed as expanding "from							243	486	[C]
the middle," named as a constant on "D.")								729	[F]

identical. However when reckoned as relative string length they differ by a barely audible amount called the Pythagorean comma. The three great means of music and mathematics are also expressed by this table. Any integer is the geometric mean of the integers on either side of it, e.g., 12 is the geometric mean of 6 and 24. Any integer is the arithmetic mean of the two integers from the row above it, e.g., 12 is the arithmetic mean of 8 and 16. Any integer is the harmonic mean of two integers from the row below it, e.g., 12 is the harmonic mean of 9 and 18. Therefore each integer serves as geometric, arithmetic and harmonic mean.

Table 2 The McClain Table

a)

25	75	135	675	2025 …	
5	15	45	135	405…	
1	3	9	27	81…	

b)

c	f	b_b	e_b	a_b
E	A	D	G	C
g♯	c♯	f♯	b	e

c)

400	600	450	675	
640	480	720/360	540	405
512	384	576	432	648

d)

0(18)	3(16)	8(14)	1(12)	6(10)
2(4)	7(2)	0(0)	5(-2)	10(-4)
6(-10)	11(-12)	4(-14)	9(-16)	2(-18)

The integers of Table 2 also represent relative string lengths. We have seen the great numerosity required by the Pythagorean scale. However, by introducing prime 5, the tones of the chromatic scale are expressed with integers limited by the octave 720/360 with the integers 360 and 720 representing the fundamental tone at D. The integers in Fig. 2a are multiples of primes 3 and 5 with powers of 3 along the rows representing the Pythagorean scale of musical fifths. Since a_b and g♯ differ by the Pythagorean comma, a_b is eliminated and the largest integer of Table 2c is then 675. All of the integers of Table 2a are then multiplied by the appropriate power of 2 to seal them within the 720/360 octave. The pitch classes corresponding to these integers are listed in Table 2b with interval number or digital root shown in Table 2d. These are the tones of the ancient Just scale. The numbers in parentheses are the deviations of these tones from the equal-tempered scale in units of cents. The tones in this system can be organized according to the vector system shown in Fig. 1. The ratios 5:6 and 3:5 represent the complementary tones, rising minor 3rds and major 6ths. The ratios 4:5 and 5:8 are major 3rds and minor 6ths, while 2:3 and 3:4 are fifths and fourths. The integers in parentheses represent the position on the tone circle corresponding to the interval. In view of this vector system the integers of the ancient musical scale are based on a balance of opposites which along with self-limitation were Plato's primary concerns.

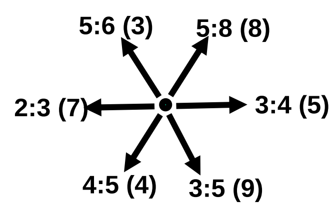

Fig. 1 *Musical vectors*

In terms of modern theories of music, Western music is built around a chord structure of the fundamental, major third, and fifth known as a *major triad* and the fundamental, minor third and fifth known as the *parallel minor triad*. Table 2b when extended, exhibits major and minor triads in every key with the major triad as the downward triangle of tones consisting of a pair of tones from row 2 bracing the tone from below in row 1 while the minor triad is the pair of tones from row 2 bracing a tone from row 3. For example, in the key of C the major triad is: C,e,G while the parallel minor is: C,e_b,G.

References

1. J. Kappraff, "The Arithmetic of Nicomachus of Gerasa and its Applications to Systems of Proportion" Nexus Network Journal (an electronic journal, www.nexusjournal.com)Vol. 4, No. 3 October 2000.
2. J. Kappraff, *Beyond Measure*, Singapore: World Scientific (2001)
3. E.G. McClain, *Myth of Invariance*, York Beach, Me.: Nicolas-Hays (1976, 1984)
4. E.G. McClain, *Pythagorean Plato*, York Beach, Me.: Nicolas-Hays (1978, 1984)