Golden Fractal Trees

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Abstract

The famed golden ratio appears in many different forms of art. Historically there have been people who believe that the golden ratio can be used to create the most aesthetically pleasing figures. More recently, fractals have been used to create beautifully intricate art. In this paper, we present an interesting connection between the golden ratio and fractal trees. Four special self-contacting fractal trees that scale with the golden ratio possess remarkable symmetries and provide new visual representations of well-known results involving the golden ratio. Golden variations of familiar fractals including the Cantor set and the Koch curve appear as subsets of golden trees.

1 Introduction

Golden Ratio. The 'golden ratio' ϕ is a remarkable number that arises in various areas of mathematics, nature and art [7], [4], [13], [3], [5], [2]. The most basic geometric description of the golden ratio involves a line segment. Without loss of generality, assume the line segment is $[0,1]$. There exists $a \in (0,1)$ such that the ratio of the length of $[0,1]$ to the length of $[0,a]$ is equal to the ratio of $[0,a]$ to $[a,1]$ (see Figure 1). By design, $a = 1/\phi$. Thus $1/\phi$ is the unique positive solution the the quadratic equation $1-x-x^2=0$, and ϕ is the unique positive solution to the quadratic equation $x^2-x-1=0$. Hence

$$\phi = \frac{1+\sqrt{5}}{2} \approx 1.618034..., \quad \frac{1}{\phi} = \frac{-1+\sqrt{5}}{2} = \phi - 1 \approx 0.618034.... \tag{1}$$

Figure 1: Dividing the unit interval according to the golden ratio

There are many interesting and useful equations that can be derived. Because $\phi + 1 = \phi^2$, any positive power of ϕ can be expressed in terms of a linear expression with ϕ. For example:

$$\phi^3 = \phi\phi^2 = \phi(1+\phi) = \phi + \phi^2 = \phi + \phi + 1 = 2\phi + 1$$

Similarly, $\phi^4 = 3\phi + 2$. For reciprocal powers of ϕ, we can also find linear equations by using $1/\phi = \phi - 1$. In general, the integer powers of ϕ can be related to the Fibonacci sequence $\{F_n\}$ ($F_0 = 1$, $F_1 = 1$, $F_2 = 2$, $F_n = F_{n-1} + F_{n-2}$ and F_n is called the nth Fibonacci number).

$$\phi^n = F_{n-1}\phi + F_{n-2}, \ \ (n \geq 2), \quad \text{and} \quad \frac{1}{\phi^n} = (-1)^n(F_n - F_{n-1}\phi), \ \ (n \geq 1) \tag{2}$$

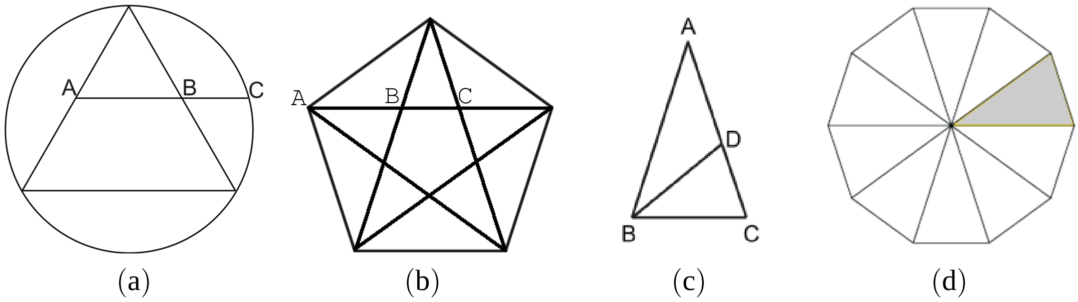


Figure 2: (a) Equilateral triangle, (b) Pentagon and Pentagram, (c) Golden Triangles ($\triangle ABC$ and $\triangle BCD$) and Golden Gnomon ($\triangle ABD$), (d) Decagon

The Fibonacci numbers F_n are connected to ϕ in yet another way:

$$\phi = \lim_{n\to\infty} \frac{F_n}{F_{n-1}}.$$

The golden ratio can be found in many classical geometrical figures. We briefly mention figures that are relevant for this paper. Consider an equilateral triangle and the circle that circumscribes it, as in Figure 2a. The line segment AB joining midpoints of two sides can be extended to the circle at C, and the ratio of AC to AB is ϕ. Now consider a pentagon and a pentagram with the same vertices as in Figure 2b. The interior angles at each vertex of a pentagon are 108°. Again one can show that the ratio of AC to AB is ϕ. A triangle with the angles $72^\circ, 72^\circ, 36^\circ$ is commonly called a *golden triangle* ($\triangle ABC$ in Figure 2c). Such a triangle can be divided into a smaller golden triangle ($\triangle BCD$) and a *golden gnomon* (a triangle with the angles $36^\circ, 36^\circ, 108^\circ$, given by $\triangle ABD$). Any triangle in Figure 2b is either a golden triangle or a golden gnomon. Golden triangles also appear in the decagon (Figure 2d).

There are many other interesting aspects of the golden ratio ϕ. The golden ratio ϕ can be considered to be the most ‘irrational’ number because it has a continued fraction representation

$$\phi = [1,1,1,\cdots] = 1 + \frac{1}{1+\frac{1}{1+\frac{1}{1+\cdots}}}. \tag{3}$$

This representation is straightforward to derive by using the equation $\phi = 1 + 1/\phi$. There is also a nested radical expression for the golden ratio which can be derived from the equation $\phi^2 = \phi + 1$:

$$\phi = \sqrt{1+\sqrt{1+\sqrt{1+\cdots}}}. \tag{4}$$

Fractals and The Golden Ratio. The golden ratio appears in fractal geometry as well as classical geometry, perhaps due to its own self-similar nature. Self-similarity is apparent in a pentagon and pentagram with the same vertices (as in Figure 2b). There is a smaller pentagon inside, and one can then construct the smaller pentagram with the same vertices as this smaller pentagon. This process can be repeated *ad infinitum*. The same idea applies to a golden triangle and dividing it into a smaller golden triangle and golden gnomon. Besides geometric representations of self-similarity, the self-similar nature of the golden ratio is also displayed in the continued fraction representation (Equation 3) and the nested radical (Equation 4).

There is no strict mathematical definition of “fractal”, but when we refer to a set F as being fractal, we generally have the following in mind:

- F has fine structure (detail at arbitrary levels)

- F is too irregular to be described in traditional geometrical language (locally and globally)
- Often there is some form of self-similarity (the whole is made up of smaller parts that are similar to the whole)
- Usually the 'fractal dimension' of F is greater than its topological dimension
- Typically F can be defined in a very simple way, possibly recursively

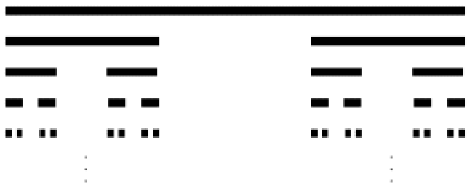
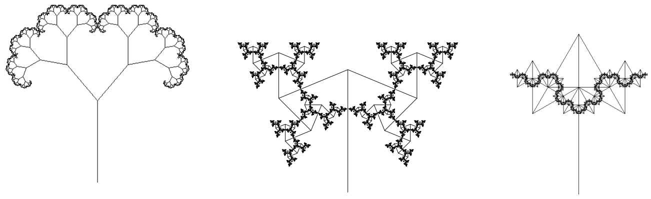


Figure 3: Middle Thirds Cantor Set

One well-known fractal is the middle thirds Cantor set, where one starts with an interval of unit length, removes the middle third, and continues the process *ad infinitum* (Figure 3). This set is self-similar, and one can find the corresponding similarity dimension D as follows. The original line segment is scaled down by a factor of 3 to obtain 2 similar line segments, so D is given by $3^D = 2$, or $D = \log 2/\log 3 \approx 0.6309$. We shall see that a 'golden' Cantor set [6] appears in several of our golden trees. The general family of middle Cantor sets $\{C_\alpha\}_{\alpha\in(0,1)}$ includes the classic middle thirds Cantor set. Given $\alpha \in (0,1)$, the middle-α Cantor set C_α is obtained from removing the middle open set of length α from the unit interval, then continuing to remove middles *ad infinitum*. It is straightforward to show that the similarity dimension of a middle-α Cantor set is $\log 2/\log(1/\beta)$, where $\beta = (1-\alpha)/2$. There are many other instances of the golden ratio and fractals appearing together in literature, and thus it is not possible to provide a complete survey here. There is an interesting relationship between the golden ratio and variations on another well-known fractal, the Sierpiński gasket [1]. Our current paper deals with the golden ratio and a specific class of fractals, the self-contacting symmetric binary fractal trees.

Fractal Trees. Fractal trees were first introduced by Mandelbrot [8]. The class of symmetric binary fractal trees were more recently studied by Mandelbrot and Frame [9]. A symmetric binary fractal tree $T(r,\theta)$ is defined by two parameters, the scaling ratio r (a real number between 0 and 1) and the branching angle θ (an angle between 0° and 180°). The trunk splits into two branches, one on the left and one on the right. Both branches have length equal to r times the length of the trunk and form an angle of θ with the affine hull of the trunk. Each of these branches splits into two more branches following the same rule, and the branching is continued *ad infinitum* to obtain the fractal tree. Each tree has left-right symmetry.



(a) $T(r_{sc}, 40^\circ)$ (b) $T(r_{sc}, 112.5^\circ)$ (c) $T(r_{sc}, 150^\circ)$

Figure 4: Self-contacting Symmetric Binary Fractal Trees

A self-contacting symmetric binary fractal tree has self-intersection but no actual branch crossings. For a given branching angle θ, there is a unique scaling ratio $r_{sc}(\theta)$ (or just r_{sc}) such that the corresponding tree is self-contacting [9]. The values of r_{sc} as a function of θ have been completely determined [9]. To determine the value of r_{sc}, find the smallest scaling ratio such that there is part of the tree besides the trunk that is on the affine hull of the trunk [9], [12]. Figure 4 displays three different self-contacting symmetric binary fractal trees. Further details about the general class of self-contacting trees are available [9], [11], [12].

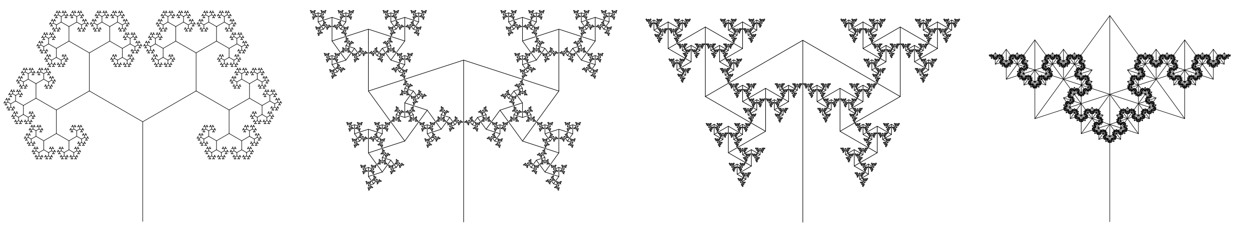
To describe the fractal trees and their scaling properties more precisely, we use an address system. For any $r \in (0,1)$ and any $\theta \in (0°, 180°)$, the *generator maps* $m_R(r,\theta)$ (or just m_R) and $m_L(r,\theta)$ (or just m_L) are:

$$m_R\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = r\begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix},\ m_L\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = r\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} \tag{5}$$

An *address* $\mathbf{A} = A_1A_2\cdots$ is a string (finite or infinite) of elements, with each element either R (for 'right') or L (for 'left'). The *level* of the address is equal to the number of elements in the string. The only level 0 address is the empty address $\mathbf{A}_0$. An address map $m_{\mathbf{A}}$ is a composition (finite or infinite) of generator maps. An infinite address map is the limit of finite address maps, and is well defined because $r \in (0,1)$. If C is any compact subset of $\mathbb{R}^2$ and if $m_{\mathbf{A}}$ is a level k address map, then the set $m_{\mathbf{A}}(C)$ is a compact subset of $\mathbb{R}^2$ that is similar to C with contraction factor r^k.

The *trunk* T_0 of any tree is the closed vertical line segment between the points $(0,0)$ and $(0,1)$. The image $m_R(T_0)$ is the first branch to the right of the trunk and the image $m_L(T_0)$ is the first branch on the left side. Given an address $\mathbf{A}$, $m_{\mathbf{A}}(T_0)$ is a branch. The *point with address* $\mathbf{A}$, $P_{\mathbf{A}}$ is $m_{\mathbf{A}}((0,1))$. If $\mathbf{A}$ is finite the point is a *vertex*, if it is infinite, the point is a *tip point*. The *symmetric binary fractal tree* $T(r,\theta)$ is defined to be the limit as the number of levels of branching goes to infinity. The *top points* of a tree are all points of the tree that have maximal y-value. For an address $\mathbf{A} = A_1A_2\cdots$, there exists a natural path on the tree that starts with the trunk and goes to $P_{\mathbf{A}}$, consisting of the trunk along with all branches $b(\mathbf{A}_i)$, where $\mathbf{A}_i = A_1\cdots A_i$. We denote this path $p(\mathbf{A})$. A *level k subtree* of a tree $T(r,\theta)$ is $m_{\mathbf{A}}(T)$ for some level k address $\mathbf{A}$, denoted by $S_{\mathbf{A}}(r,\theta)$ or $S_{\mathbf{A}}$.

2 Golden Trees



(a)$T(1/\phi, 60°)$ (b)$T(1/\phi, 108°)$ (c)$T(1/\phi, 120°)$ (d)$T(1/\phi, 144°)$

Figure 5: Golden Trees

We now discuss the relationship between the golden ratio and self-contacting symmetric binary fractal trees. There are exactly four self-contacting symmetric binary fractal trees for which $r_{sc} = 1/\phi$ (see Figure 5), corresponding to 60°, 108°, 120° and 144° [11]. The golden tree $T(r_{sc}, 60°)$ has been discussed in other literature [13], [10]. Each tree possesses remarkable symmetries in addition to the usual left-right symmetry of symmetric binary trees. The trees all seem to 'line up'. Most of the observations and figures are given without proof, and generally the proofs are wonderful exercises involving trigonometry, geometric series, the scaling nature of the fractal trees, and the many special equations involving the golden ratio (so very good for students!).

To determine r_{sc} for any angle, we find a path which leads to a point on the subtree S_R that has minimal distance to the y-axis (other than the point that is the top of the trunk). For $T(r_{sc}, 60°)$, the path is $p(RL^3(RL)^\infty)$, for both $T(r_{sc}, 108°)$ and $T(r_{sc}, 120°)$, the path is $p(RR(RL)^\infty)$, and finally, the path for $T(r_{sc}, 144°)$ is $p(RR)$. For any self-contacting tree, a *self-contact point* refers to a point on the tree that is on the x-axis that corresponds to more than one address. For the tree $T(1/\phi, 60°)$, a self-contact point is $P_{RL^3(RL)^\infty}$, because this point is the same as $P_{LR^3(LR)^\infty}$. The reader can verify that $r_{sc} = 1/\phi$ for each angle.

The top points of the golden tree $T(1/\phi, 60°)$ form a golden middle Cantor set. Let l be the length of the top of the tree, *i.e.*, the length between the leftmost top point $P_{(LR)^\infty}$ and the rightmost top point $P_{(RL)^\infty}$. The top points are geometrically similar (with a factor of $\sqrt{3}$) to the middle Cantor set with $\alpha = 1/\phi^3$. The first two iterations of removing open middles are shown in Figure 6. At the first iteration, the two remaining line segments each have length l/ϕ^2, because they are on the tops of level 2 subtrees S_{RL} and S_{LR}. The length of the gap at the first iteration must be equal to l/ϕ^3. Thus $\frac{l}{\phi^2} + \frac{l}{\phi^3} + \frac{l}{\phi^2} = l$. Simplifying this expression gives $2\phi + 1 = \phi^3$, as established in Equations 1. Thus we have a visual representation of one of the many equations involving ϕ. The scaling dimension of this golden Cantor set is $\log 2/(2\log\phi)$. The top of the tree $T(1/\phi, 60°)$ yields a 'golden' Koch curve (see Figure 7).

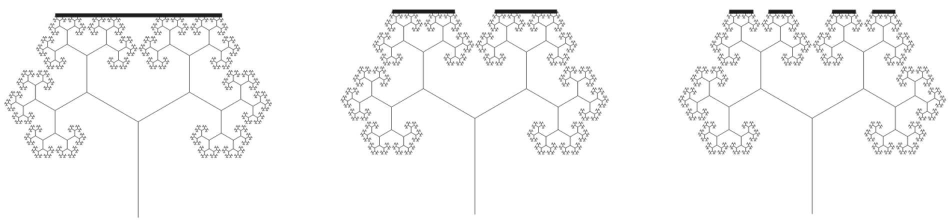


Figure 6: Golden Cantor Set on $T(1/\phi, 60°)$

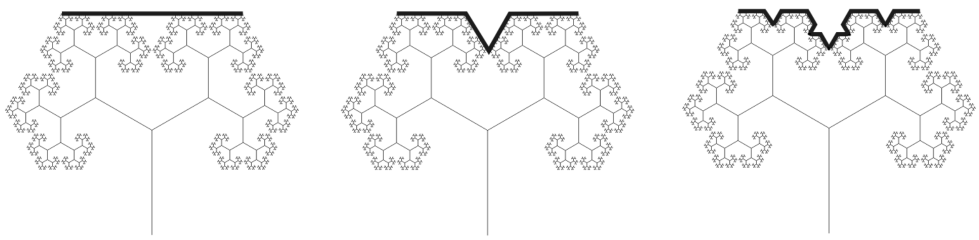
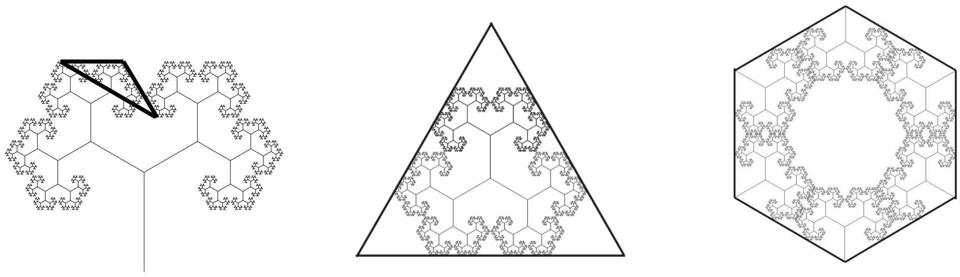


Figure 7: Golden Koch Curve on $T(1/\phi, 60°)$

Another well-known equation involving ϕ can be visualized on $T(1/\phi, 60°)$. Because $\theta = 60°$, the tops of the subtrees S_{LRR} and S_{RLLL} are collinear. Consider the triangle consisting of the top of the subtree S_{LR} as one side, the tops of S_{LRR} and S_{RLLL} as another side, and the line segment joining them as in Figure 8a. This triangle is an isosceles triangle because the angles are $120°, 30°, 30°$. Setting the sides equal and using the scaling nature of the tree gives $\frac{l}{\phi^2} = \frac{l}{\phi^3} + \frac{l}{\phi^4}$. This equation simplifies to the familiar equation $\phi^2 = \phi + 1$.

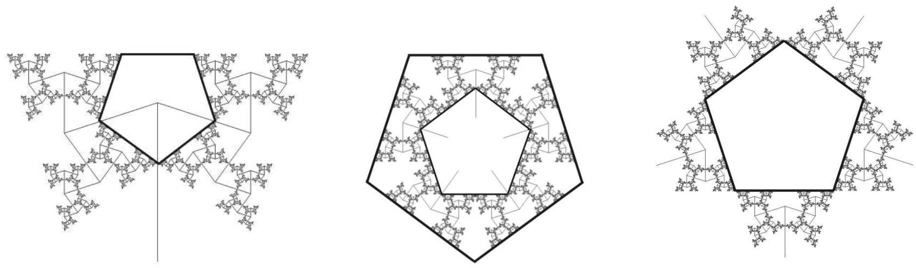
Equilateral triangles and hexagons can also be associated with $T(1/\phi, 60°)$, as in Figures 8b and c. The boundary of the center region in Figure 8c is a golden Koch snowflake .

As with $T(1/\phi, 60°)$, the top points of $T(1/\phi, 108°)$ form a golden Cantor set. Let l be the length of the top of the tree, *i.e.*, the distance between $P_{(LR)^\infty}$ and $P_{(RL)^\infty}$, then the gap between $P_{LR(RL)^\infty}$ and $P_{RL(LR)^\infty}$ has length l/ϕ^3. One can form a pentagon from the tops of the level 3 subtrees S_{LRR}, S_{LLL}, S_{RRR} and S_{RLL} along with the line segment in the gap (as shown in Figure 9a). The interior angles at each vertex of a pentagon are



(a) (b) (c)

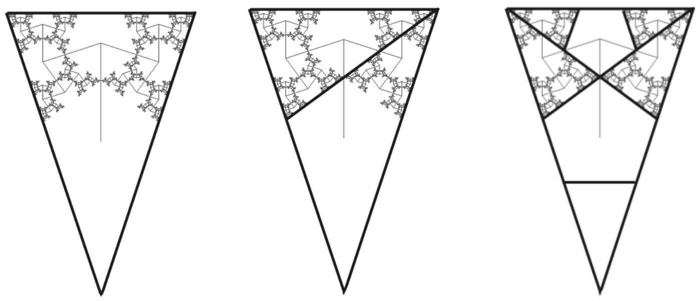
Figure 8: Triangles associated with $T(1/\phi, 60°)$



(a) (b) (c)

Figure 9: Pentagons associated with $T(1/\phi, 108°)$

108°, thus it seems natural that pentagons, and in turn the golden ratio, are relevant for this tree. Figures 9b and c show other related pentagons. Golden triangles, golden gnomons and pentagons can be seen in Figures 10a,b and c.

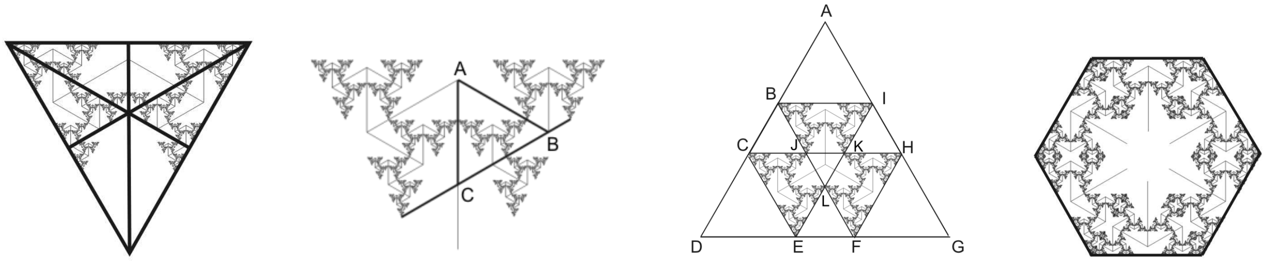


(a) (b) (c)

Figure 10: Golden triangles, golden gnomons and pentagon in $T(1/\phi, 108°)$

As with $T(1/\phi, 60°)$ and $T(1/\phi, 108°)$, the top tip points of $T(1/\phi, 120°)$ form a golden Cantor set. With branching angle 120°, it is not surprising that there are equilateral triangles and hexagons associated with the tree. One equilateral triangle is shown in Figure 11a. Another equilateral triangle is displayed in Figure 11b. This triangle consists of the branch $b(R)$ (segment AB), the extension of the branch $b(RR)$ to the trunk (segment BC), and the portion of the trunk that joins these two line segments (segment CA). The length of each side is $1/\phi$, so the line segment that extends from $b(RR)$ divides the trunk according to the golden ratio. This same figure also shows that the point C divides the trunk according to the golden ratio (because

AC has the same length as *AB*).

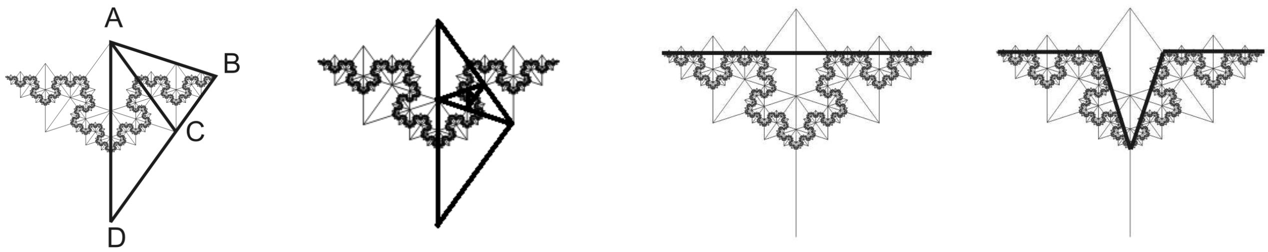


(a) (b) (c) (d)

Figure 11: $T(1/\phi, 120°)$: collinearity, equilateral triangles and hexagon

Figure 11c displays other equilateral triangles related to the triple tree. $\triangle ABI$ is equilateral, and each side of the triangle is *l* (where *l* denotes the length of the top of the tree). Likewise for $\triangle CDE$ and $\triangle HFG$. $\triangle BCJ$ is equilateral, with sides of length l/ϕ. Thus the line segment *AD* has length $l + l/\phi + l$. One can show that the ratio of *AD* to *AC*is ϕ. Thus triangles $\triangle ACH$, $\triangle BDF$ and $\triangle IEG$ are similar to $\triangle ADG$ with factor $1/\phi$. The iterated function system which consists of the three similarities that send the largest equilateral triangle $\triangle ADG$ to three triangles scaled by $1/\phi$, namely $\triangle ACH$, $\triangle BDF$ and $\triangle IEG$, corresponds to a variation on the usual Sierpiński gasket that is also self-similar [1].

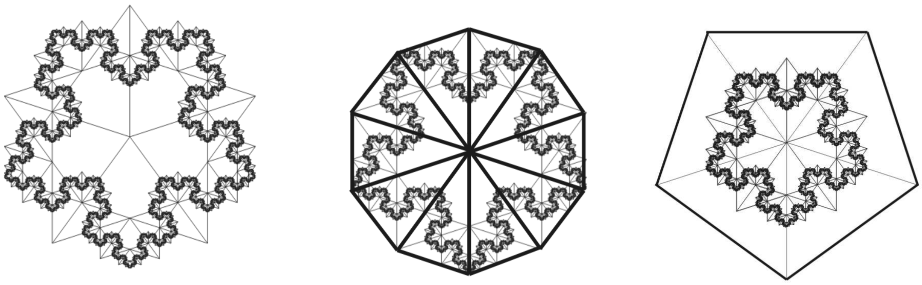
Finally, the last tree we consider is $T(1/\phi, 144°)$. This angle is supplementary to 36°, so it is natural that golden triangles are found. In Figure 12a. The line segment *BD* through the origin (at *D*) and the point with address *R* (at *C*) goes through all points of the form $P_{R(LR)^k}$ for $k \geq 0$ and $P_{R(LR)^\infty}$. This line segment *BD* has length 1. The line segment *AB* goes through the all points of the form $P_{(RL)^k}$ for $k \geq 0$ and $P_{(RL)^\infty}$. The triangle $\triangle ABD$ is a golden triangle. The line segment *AC* that goes from the top of the trunk to the point P_R divides $\triangle ABD$ into a golden gnomon ($\triangle ACD$) and a smaller golden triangle ($\triangle ABC$). Figure 12b shows branches of the form R^k. They form a spiral of golden triangles.



(a) (b) (c) (d)

Figure 12: Golden figures on $T(1/\phi, 144°)$

The tip points of $T(1/\phi, 144°)$ form a golden Koch curve. The initial line segment is shown in Figure 12c, with the first iteration shown in Figure 12d. A line segment of length *l* is replaced with four line segments each of length l/ϕ^2. Figure 13a shows the tree along with four copies, rotated around the bottom of the trunk with angles that are multiples of 72°. The tip points of the trees form a golden Koch snowflake. Figure 13b shows a decagon associated with the tree and four copies. Finally, Figure 13 shows a pentagon with each vertex acting as the bottom of a trunk of a tree and such that the five trees have the share the same point as the top of the trunk.



(a) Golden Koch Snowflake (b) Decagon (c) Pentagon

Figure 13: More golden figures related to $T(1/\phi, 144°)$

3 Conclusions

This paper has presented four self-contacting symmetric binary fractal trees that scale with the golden ratio. The four possible branching angles are 60°, 108°, 120°, and 144°. Various geometrical figures, other fractals, and equations associated with the golden ratio can be associated with these special trees. The author hopes that the readers have enjoyed new ways to see the symmetry and beauty of the golden ratio and fractals.

ACKNOWLEDGEMENTS

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