Musical Scales, Integer Partitions, Necklaces, and Polygons

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Abstract

A musical scale can be viewed as a subsequence of notes taken from a chromatic sequence. Given integers (N, K) $N > K$ we use particular integer partitions of N into K parts to construct distinguished scales. We show that a natural geometric realization of these scales results in maximal polygons.

1 Introduction

In his book on jazz ear training Steve Masakowski [6] speaks of four key scales that form the basis of organization that allows jazz musicians to understand and follow the harmony and melody of a piece. In his paper on seven tone collections Jay Rahn [7] analyzes seven note collections from the perspective of interval structure. With a small exception his seven note collection is identical to the Masakowski collection. In this note we present a combinatorial approach to generating a seven note collection of scales. The collection is not derived from musical consideration but is a collection that satisfies a geometric constraint. Again this collection with some small exceptions matches those of the Masakowski and Rahn collections.

A landmark result in combinatorial music theory is the characterization of the diatonic scale by Clough and Douthett [1]. The characterization can be made in numerous ways. In a subsequent paper Clough et al. [2] enumerate eight different characterizations of the diatonic scale. A more recent mathematical treatment looking at many of these results pertaining to rhythms as well as scales can be found in the paper by Demaine et al. [3] . If we consider a scale as a subsequence of a chromatic circular sequence to be realized by a set of equally spaced points on the circumference of a unit circle, then we can express the distance between notes of the scale as the Euclidean distance between their point representations. Using this representation there is a simple geometric property that encapsulates all of these rules and is unique to the diatonic scale. The diatonic scale is the unique seven note scale (up to rotations) that maximizes the sum of Euclidean distances between every pair of notes.

In this note we take a purely mathematical approach at deriving an interesting group of seven note collections. Rather than base the choice on any deep musical analysis, we will simply define some mathematical objective, solve for it, and report the results. The underlying contribution that is made is a distillation of a plethora of rules and properties into a simple and general framework. Just as the diatonic scale is characterized by a single geometric property, the collection of seven note scales presented are all maximal in a geometric sense.

2 Preliminaries

A musical scale can be viewed as a subsequence of notes taken from a chromatic sequence. Thus we can use the notation (N, K) scales to denote scales of K notes taken from an N note chromatic universe. We develop some notation to describe (N, K) scales in general before we turn our focus to (12,7) scales in particular.

A well known combinatorial object, the "necklace", captures the notion of modelling a scale with a circular sequence. A n-ary necklace is defined as an equivalence class of n-ary strings under rotation, see [10]. More intuitively, think of a string of beads. The string of beads is an implicit sequence that is invariant under rotations. Relating this concept to the diatonic scale consider a string of black and white beads arranged in the same pattern as 12 white and black keys comprising a single octave on the piano. Note that in his well known list of scales Alan Forte [5] considers equivalence classes of strings that are invariant under rotation and reflection. In music theory terminology rotation = transposition and reflection = inversion. The requisite combinatorial structure that can be used to enumerate sequences that are invariant under rotation and reflection is the *bracelet*. Much is known about the cardinality of necklaces and bracelets, and both of these combinatorial objects can be enumerated with very efficient algorithms, see [10, 9].

An additional property that is needed to perform our analysis is to embed the combinatorial necklace onto a concrete geometric surface. Thus we can consider the beads to be placed at equally spaced intervals on the circumference of a circle of radius one.

For an n-ary necklace we can label the beads from $0 \dots n-1$, and by convention always assume that the 0 bead is white. This labelling is useful for describing the distance between any two white beads. We use three distinct distances. Thus for two white beads $i, j \; i \neq j$, we have:

chromatic The chromatic distance between any two distinct white beads i and j is denoted by $c_{i,j} = |\{k : k \in [i+1 \dots j]\}|$ that is the total number of beads in the substring $[i+1 \dots j]$.

scalar The scalar distance between any two distinct white beads i and j is given by $d_{i,j} = |\{k : k \in [i+1 \dots j]$ and k is white$\}|$ that is the total number of white beads in the substring $[i+1 \dots j]$.

Euclidean The Euclidean distance between any two distinct white beads i and j is denoted by $e_{i,j}$ is the length of the line segment between points representing the beads.

In 1956 Fejes Tóth asked for the configuration of points on a continuous circle of fixed radius that maximizes the sum of inter-point distances. The answer is to place the n points at the vertices of a regular n-gon [4]. Asking a similar question for placing points on a discrete circle, that is a circle with a finite number of equally spaced possible positions with fixed radius, yields the diatonic collection as the unique answer when the number of points is seven and the number of positions on the circle is twelve. In essence the points are spread out as evenly as is possible. Hence the name given by Clough and Douthett *maximally even* [1].

The quantity that is maximized can be written as:

$$\sum_{\text{for all distinct i,j}} e_{i,j}.$$

This sum can be broken down into components, each representing the sum of the inter-point distances for pairs that are at the same scalar distance. Let

$$\Sigma_\ell = \sum_{i,j:d_{i,j}=\ell} e_{i,j}$$

Our distinguished set of scales are defined as those scales whose point representation maximizes individual Σ_ℓ quantities. Noting that $\Sigma_i = \Sigma_{K-i}$ for $i = 1 \dots \lfloor \frac{N}{2} \rfloor$ we consider those scales that maximize $\Sigma_1, \Sigma_2, \dots, \Sigma_{\lfloor \frac{N}{2} \rfloor}$.

We now present a general and efficient way to obtain this collection of scales.

An *integer partition* of a natural number N is a way of writing N as an unordered sum of natural numbers. Consider positive integers $N, K \; K < N$ and a partition of N using exactly K positive integer summands, that is, $N = a_1 + a_2 + \cdots + a_K$. The values $a_i, 1 \leq i \leq K$ denote the different chromatic distances between notes that are at scalar distance 1, or in musical terminology the length of an interval of a second. In [8] the scales with the property that notes i, j at scalar distance $d_{i,j} = 1$ have their chromatic distance $c_{i,j}$ come from two consecutive values that differ by at most one are examined. It was shown that given integers N, K with $K < N$, there exists a unique m such that $N = m \left\lfloor \frac{N}{K} \right\rfloor + (K - m) \left\lceil \frac{N}{K} \right\rceil$.

Thus let an integer partition of N into K summands $a_i, 1 \le i \le k$ such that $a_i \in \{\left\lfloor \frac{N}{K} \right\rfloor, \left\lceil \frac{N}{K} \right\rceil\}$ be called an *even integer partition* of N into K summands, which we denote by EP(N, K).

In [8] it is shown that any scale (N,K) with the property that notes i, j at scalar distance $d_{i,j} = 1$ have chromatic distance $c_{i,j} \in$ EP(N,K) maximizes area. A similar approach can be used to show that the sum of distances between adjacent points is maximized.

As integer partitions are not ordered and our model of a scale is, we need to impose a particular ordering of the summands to obtain a scale. For example consider the values $N = 12$ and $K = 7$ there are three distinct scales resulting from EP(12,7). The method to enumerate these scales is by using combinatorial necklaces. Rather that use necklaces with N beads, K of them white, we use K beads where m are labelled $\left\lfloor \frac{N}{K} \right\rfloor$ and $K - m$ are labelled $\left\lceil \frac{N}{K} \right\rceil$. The distinct number of these necklaces enumerates the different scales with this property. It should be noted that enumerating all two coloured fixed density necklaces (that is we fix the number of beads of each type) can be performed in time that is a linear function of the total number of necklaces enumerated [10].

3 Necklaces and Polygons

For melodic considerations it is desirable to have “smooth” diatonic steps. Thus, as was discussed above, we enumerate the necklaces obtained from EP(12,7).

The triad is the basic building block in Western harmony. A triad consists of three notes and there is an interval of a third between the first and second and the second and third notes. The interval of a third is made up of 3 or 4 chromatic steps. If we think of traversing a scale by leaps of consecutive thirds, that is by skipping over one note, we see that we make two complete revolutions around the scale. This is the intuition that explains why we consider an even integer partition EP(24,7). Observe that $\left\lfloor \frac{24}{7} \right\rfloor = 3$, and that $\left\lceil \frac{24}{7} \right\rceil = 4$, and $24 = 3*4+4*3$. Thus we consider distinct necklaces that can be obtained with seven beads where three are labelled 4, and four are labelled 3.

One of the attributes of the Diatonic scale is that it can be obtained using a generator, see [2]. The next necklace we consider represents the even integer partition EP(36,7). We see that 36 = 6*5+6, or expressed in another way $36 \equiv 1 \pmod 7$. This in turn implies there is a unique necklace corresponding to EP(36,7). Observe that the sequence of diatonic notes taken at chromatic distance 5 corresponds to the familiar *circle of fourths*.

	Interval	Pitch	Notes	Forte No.	M	R
1	1221222	0, 1, 3, 5, 6, 8, 10	Diatonic	7-35	Y	Y
2	1212222	0, 1, 3, 4, 6, 8, 10	Melodic Minor	7-34	Y	Y
3	1122222	0, 1, 2, 4, 6, 8, 10	Neapolitan	7-33	N	Y
4	3343434	0, 1, 3, 5, 6, 8, 10	Diatonic	7-35	Y	Y
5	3343344	0, 1, 3, 4, 6, 8, 10	Melodic Minor	7-34	Y	Y
6	3334434	0, 1, 4, 5, 7, 9, 10	Harmonic Major	7-32	Y	Implicit
7	3334344	0, 2, 3, 5, 7, 8, 11	Harmonic Minor	7-32	Y	Y
8	3333444	0, 1, 3, 5, 6, 9	Aug. Triad & Dim.7	6-Z28	N	N
9	5555556	0, 1, 3, 5, 6, 8, 10	Diatonic	7-35	Y	Y

Table 1: Necklaces and scales corresponding to EP(12,7) [1...3] , EP(24,7) [4 ...8] and EP(36,7) [9] .

We enumerate the appropriate necklaces and present the results in Table 1. We have identified each scale by an interval sequence. This corresponds directly to the enumerated necklaces. Each interval sequence is in turn represented by a pitch sequence. This pitch sequence can be interpreted either by fixing 0 to C, or more neutrally to a moveable "Do" system where 0 corresponds to scale degree $\hat{1}$. We then use a familiar name to identify the scales. We obtain five uniquely named 7 note scales, diatonic, melodic minor (ascending), neapolitan (or whole-tone plus a note), harmonic major, and harmonic minor. Note that our analysis produces one 6 note scale, which may be described as an augmented triad superimposed with a symmetric diminished seventh chord. The reason we only get 6 notes is revealed by the interval sequence 3333444. The sequence of four consecutive three's produces a second copy of the 0 pitch. We also use Forte's scale numbering system, [5] as one further standardized representation. Note that the harmonic major and harmonic minor scales share the same Forte number, because one is just a reflection of the other. In the final two columns of the table we denote the scales that are in the Masakowski [6] and Rahn [7] collections respectively. All of our seven note scales appear in the Rahn collection, although the harmonic major is implicit as it is the reflection of the harmonic minor. Only the Neapolitan scale is missing in the Masakowski list.

The necklaces in all but one case lead to a 7-gon inscribed in a regular 12-gon, 24-gon, or 36-gon. Each of the inscribed 7-gons are maximal polygons, that is they maximize either $\Sigma_1, \Sigma_2,$ or Σ_3. The diatonic scale is unique as it maximizes all three. Also note that the six note scale that is obtained is an anomaly. Not only does it have less that seven notes, it is not maximal for 6-gons.

4 Discussion

We have shown in one unifying method a way to characterize a collection of seven note scales. The collection is constructed by a simple enumeration of scales that satisfy a geometric property. This collection matches collections that are chosen for conforming to detailed harmonic and melodic considerations. Thus complex properties involving intricate structures are distilled into a simple mathematical formula.

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