Knot Designs from Snowflake Curves

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Abstract

The Koch snowflake curve is one of the best-known self-similar fractals. Natural modifications of the polygons that represent the early stages of its generation provide templates for knotwork designs, some of which have been used in bookbindings. The boundary of another well-known self-similar fractal, the Sierpinski gasket, is closely related, and suggests a way to construct fractal knots.

Snowflake Curves

In 1904 Helge von Koch described a curve that has no tangent at any point, encloses a finite area, but the length of the curve between any two of its points is infinite. Start with an equilateral triangle and replace the middle third of every line-segment with two sides of an outward pointing equilateral triangle. Apply this rule to every line-segment, including the ones generated by its application. The curve is generally seen as the limit of a series of polygons, each produced from the previous one by a single application of the replacement rule (figure 1).

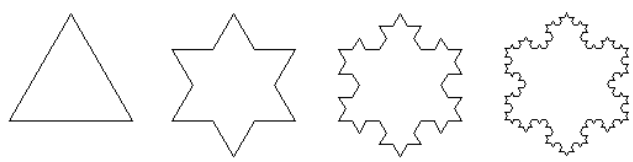

Figure 1: *The first four stages in the development of a Koch snowflake curve.*

The second stage is easily seen as the outline of the polygon (6/2), or Star of David, and this elaboration can extend to further stages in a natural way (figure 2). There are many ways to describe this sequence, for example the Star of David consists of six equilateral triangles joined at their corners to form a ring. At each step make the next in the series by overlaying each small triangle with a copy rotated by 60°.

Another two descriptions of the series are easier to generalize, and will be used later. The first stage consists of n = 6 triangles joined at their corners in a ring. A later stage consists of n = 6 copies of the previous stage joined at their corners in a ring. A scale factor is needed (in this case 1/3) to keep a constant overall size.

Notice that the Star of David has lines parallel in pairs. Consider, for example, the horizontal pair. Make a smaller copy so that its top corner coincides with the top corner of the original, and its *lower* horizontal lies on the *upper* horizontal of the original. Further stages follow the same pattern, but there are many parallels to choose from, so we need to specify corresponding pairs: corresponding parallels are related by a half-turn about the centre of symmetry

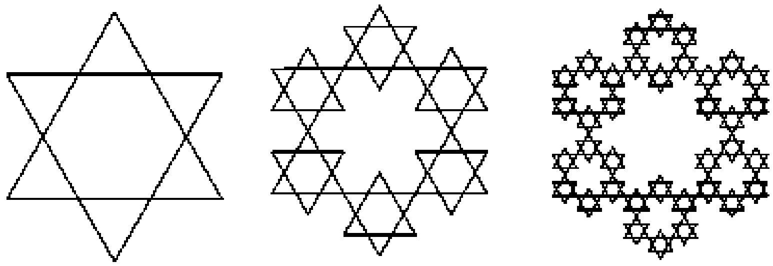

Figure 2: *Elaborating the stages in the development of a Koch snowflake curve.*

Knotwork Designs

Knotwork is a common feature of ornament in almost all cultures, reaching its most elaborate forms in the British Isles in the centuries before the Norman conquest, and in Islamic culture throughout its history. The designs to be derived from snowflake curves are different from traditional patterns, but inevitably they share some characteristics, being generally symmetrical and relying on the illusion of interlacing for their visual impact.

While it is no means an unbreakable rule there seems to be some preference in traditional examples for designs that are unicursal and do not break into disjoint segments. With this restriction, and a requirement that the original pattern in its entirety should form most of the structure of the final design, the number of possibilities becomes manageable, at least in the simpler cases.

The Star of David has three sets of parallel line-segments, and in the first stage of figure 2 they are all of the same type. In the second stage there are three types: a long segment inherited from the first stage, a short segment, its image, and a broken segment, which is nearest the centre. Any symmetric unicursal interlacing must cycle through these three types, although clearly taken from different parallel sets. The end of a cycle must be a rotation by 60° of the start. If it were 120° or 180° then there would be three or two disjoint paths respectively. Changes of direction should not occur too far out, or the original pattern would tend to insignificance, nor within the original line-segments, or some of the original pattern would be lost. Figure 3 shows two possibilities fulfilling these conditions. In both cases an additional, fourth type of line is needed.

The next stage has nine types of line, providing a multitude of possibilities.

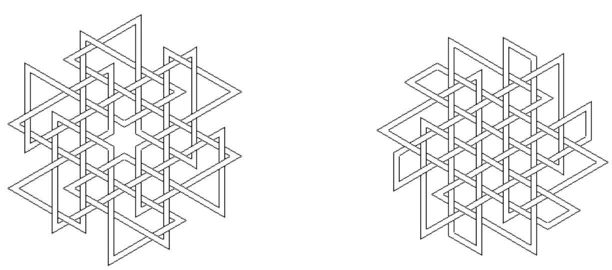

Figure 3: *Two knotwork designs derived from the second stage in figure 2.*

Some Designs Used on Books

Knot designs can be based on other polygons than (6/2) by applying the principles described earlier. Clearly any polygon can be used to form a ring by joining copies at their corners, but the second method, fitting smaller copies into an enclosed region, will only work with star polygons with an even number of sides. Figure 4 shows the first stages if the polygon (8/3) is used.

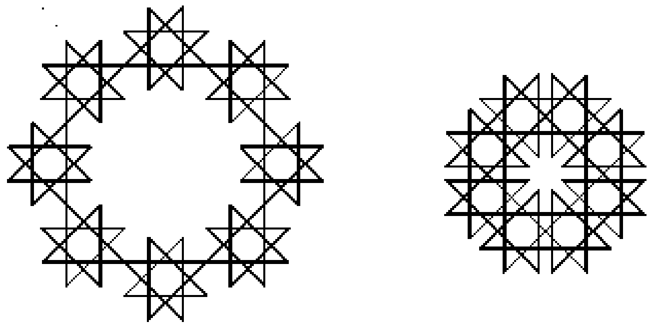

Figure 4: *The two methods give different results with (8/3).*

The second method gives a much more compact pattern, which I have used in the design of a bookbinding (figure 5). I decided that a non-unicursal interlacing is visually more interesting in this case. It makes obvious the relation between the two sizes of (8/3), and I have used colour to emphasise the difference. The book is about Tunisia, so I have chosen a design that relates strongly to traditional Islamic decorations.

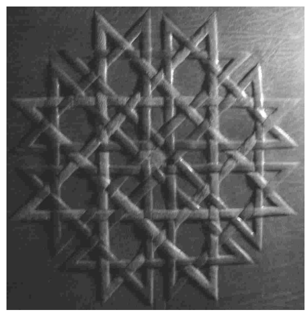

Figure 5: *A bookbinding design based on (8/3)*

A more intricate design is based on (10/4). This polygon is a compound of two pentagrams, (5/2), and single copies of (5/2) have been fitted into the (10/4) (figure 6), rather than complete (10/4)s, which would make things unduly complicated.

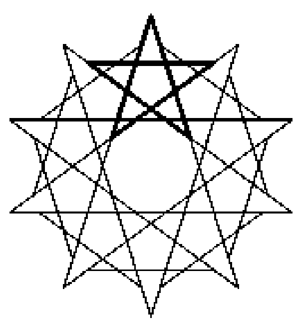

Figure 6: *Pentagrams fitted into (10/4)*

There are only two types of line, making a unicursal pattern almost impossible, but a third type arises naturally by completing small pentagrams at the outer corners. Figure 7 shows the finished design as it appears on the book. It has been adjusted to allow a greater width of straps, so pentagrams such as the one marked in figure 6 have been distorted slightly. The book is *Four Quartets* by T.S.Eliot.

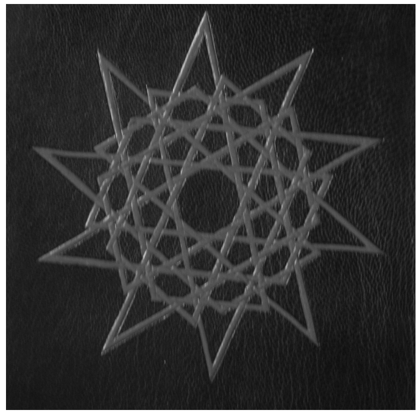

Figure 7: *A bookbinding design based on figure 6.*

A Fractal Knot

All of the examples considered so far have been based on the second stage in the construction of a snowflake curve. Later stages get increasingly complicated, and each one would need a different method to complete a unicursal knotwork design. In order to continue indefinitely each new stage must follow automatically from the previous one, but in all the designs so far the parts added to the underlying structure interfere with each other. The alternative, joining polygons at their corners, provides a workable method, but with an even number of polygons the interlacing cannot be unicursal. The simplest polygon with an odd number of sides is obviously a triangle, and this produces a series that forms the boundary of the series used to generate the Sierpinski gasket: start with a triangle (usually equilateral), join the mid-points of edges to divide into four congruent copies and remove the middle one, repeat with the three remaining, and so on (figure 8).

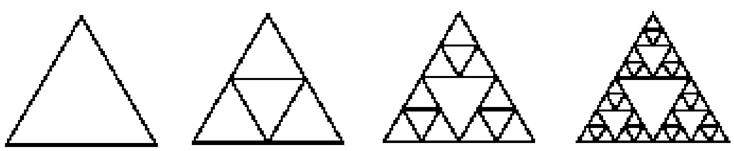

Figure 8: *The first four stages in the development of the boundary of the Sierpinski gasket.*

There are several difficulties in convert this to a knotwork design. Most obviously there is a change of direction at the point where the paths cross. This can be resolved by truncating the triangles to hexagons (figure 9), but considerable care is needed to ensure that each element has the right parity (whether the lacing goes under or over). Nevertheless there is a recursive procedure that ensures that each stage leads correctly to the next.

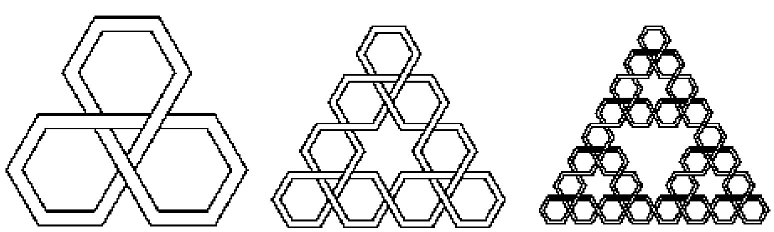

Figure 9: *The first three stages in the generation of a fractal knot.*