A Method for Illustrating Border and Wallpaper Patterns

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Abstract

Illustrations were designed to help students produce their own examples of the 7 border symmetry groups and the 17 wallpaper symmetry groups. The images attempt to guide the viewer through their stages of development, from simple to complicated. The intention is to allow one to see, step by step, how each drawing was made with the use of a base grid. A student can draw examples of all of the border and wallpaper groups with the use of just two grids: the standard square grid and the isometric grid.

1. Inspiration and Background

Artists as early as the Moors in 11th century Spain represented the symmetry groups, but mathematicians were relatively slow to identify them. In fact, mathematicians only recognized the one- and two-dimensional symmetry groups by way of the 230 three-dimensional crystallographic groups, first enumerated by E. S. Fedorov in 1885. Fedorov also appears to be the first to enumerate the 17 wallpaper groups in 1891. P. Niggli was the first to enumerate the seven border (i.e., frieze) groups in 1926. These groups and the patterns that represent them have fascinated a myriad of authors and artists ever since [2].

In preparation for a new course in *Mathematics and Visual Art*, I designed illustrations to help my students draw their own examples of the seven border (one-dimensional) symmetry patterns and the 17 wallpaper (two-dimensional) symmetry patterns. Certainly, with the aid of a computer and software such as *KaleidoMania!* by Key Curriculum Press or *Tess* by Pedagoguery Software, one can easily generate examples of any pre-programmed symmetry patterns. The software user can simply select a symmetry group, draw a scribble, and the computer will automatically perform the needed isometries to generate the selected pattern. However, students of symmetry may develop a more robust understanding of the patterns and their generating isometries if they learn steps for drawing the designs by hand. As a scaffold for this learning, this paper presents a step-by-step method for illustrating the symmetry patterns.

The method is designed to address one big obstacle: many people have difficulty organizing their hand drawings for even simple periodic patterns. So the method starts with different types of graph papers to allow students to more easily arrange their doodles into patterns. To model this technique, I developed a complete set of illustrations to guide students in drawing their own examples of the symmetry groups. This paper provides a subset of these illustrations.

The artist M. C. Escher inspired the style of these drawings. Escher writes of his artwork, "In *Regular Division of the Plane 1*, a process of development takes place. The viewer is invited to follow this by going through the phases, one after the other, of a band of images that… fills the image plane… through… stages of growth and metamorphosis…" [1]. Similarly, the images herein attempt to guide the viewer through their stages of development, from simple to complicated, or from one symmetry pattern to another. The intention is to allow students to see, step by step, how to draw each pattern.

2. Mathematics Useful for Illustrating Symmetry Patterns

A few terms are necessary for describing the illustrations of the border and wallpaper symmetry patterns; so these terms are briefly described below. This paper does not provide a comprehensive treatment of symmetry patterns. For more details on the mathematics of symmetry patterns, see [3].

2.1 Isometry. An isometry is a rigid transformation that preserves distance between points. The four isometries for the border and wallpaper patterns are rotation, reflection, glide reflection, and translation. Every border pattern has a translation that is repeated infinitely many times in two opposite directions. We say a border pattern remains unchanged when translated by a translation vector and any integral multiple of the vector. In contrast, wallpaper patterns repeat in at least two different directions. We say that all wallpaper patterns remain unchanged when translated by two nonparallel vectors of translation and any integral linear combinations of those vectors. The pattern type determines the possible angles between the two vectors.

An important graphic property of the four isometries is that all isometries are products of reflections. This is important to an illustrator because the number of reflections is related to handedness. In particular, translations and rotations preserve handedness; so they must be the product of an even number of reflections. Reflections and glide reflections reverse handedness; so they are products of an odd number of reflections.

2.2 Cyclic Symmetry. Since cyclic symmetries include only rotations, designs with cyclic symmetry often resemble pinwheels. By definition, the set of clockwise rotations r_k around a fixed center point C is a cyclic group of order n, denoted $\boldsymbol{C}_n$ where the $r_k = 360k/n$ are measured in degrees and $0 \leq k < n$. If a pattern has a rotational symmetry, then it can be rotated about a point by a fixed angle, and the pattern appears unchanged. For example, if an equilateral triangle rotates 120º about its center, then it would appear unchanged. We say that an equilateral triangle has 120º-rotational symmetry, also called $\boldsymbol{C}_3$ symmetry. All designs have trivial 360º-rotational (i.e., $\boldsymbol{C}_1$) symmetry, so we say a pattern with only 360º-rotational symmetry has no nontrivial rotations. The only nontrivial rotational symmetry that border patterns can have is 180º or $\boldsymbol{C}_2$ symmetry. Those for wallpaper patterns are $\boldsymbol{C}_2$, $\boldsymbol{C}_3$, $\boldsymbol{C}_4$, and $\boldsymbol{C}_6$ symmetry in various combinations.

2.3 Dihedral Symmetry. Dihedral symmetries include both reflections and rotations. When a symmetric pattern is flipped over a line of reflection, the pattern will appear unchanged.

By definition, a dihedral group $\boldsymbol{D}_n$ contains the n elements of the cyclic group of rotations r_k together with n reflections through n lines in the plane, all of which intersect at the center of rotation C, and the angles formed by the intersecting lines are the r_k. A pattern that appears unchanged under the isometries of a dihedral group is said to have dihedral symmetry. For example, since an equilateral triangle has three lines of reflection that all intersect at the center of rotation, an equilateral triangle has $\boldsymbol{D}_3$ symmetry. Similarly, a non-square rectangle has $\boldsymbol{D}_2$ symmetry. $\boldsymbol{D}_1$ symmetry consists of only a single reflection and only a trivial rotation. The only dihedral symmetry that border patterns can have is $\boldsymbol{D}_1$ and $\boldsymbol{D}_2$. Those for wallpaper patterns are $\boldsymbol{D}_1$, $\boldsymbol{D}_2$, $\boldsymbol{D}_3$, $\boldsymbol{D}_4$, and $\boldsymbol{D}_6$ symmetry in various combinations.

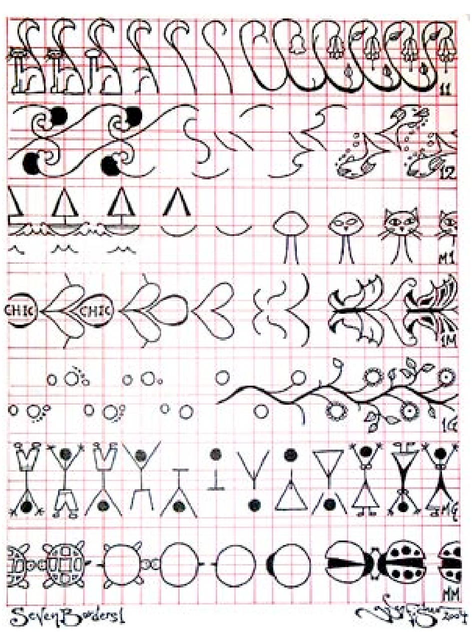
2.4 Grids. All of the border and wallpaper groups are illustrated with the use of just two grids: the square grid (i.e., the regular tiling by squares) and isometric grid (i.e., the regular tiling by equilateral triangles). Only two types of grids are needed because each of the seven border and 17 wallpaper symmetry groups is a subgroup of the symmetry group of at least one of these two grids. For artistic purposes, I used a third grid to draw the border patterns. This grid is a simple modification of a square grid and reminds me of the paper on which I practiced my handwriting when I was a child. One could also use any fixed number of rows on a square grid or on a triangular grid to draw the border patterns.

2.5 A Note on Mathematical Accuracy. It should be stated explicitly that the illustrations in this paper are not mathematically accurate representations of border and wallpaper patterns, strictly speaking. The patterns herein change subtly as one moves across the pattern. In contrast, mathematical accuracy requires that a border or wallpaper pattern repeat in exactly the same way, forever, showing infinitely many repeats. The bounds of a sheet of paper, "a fragment of a plane," as Escher called it, make the mathematical abstraction of infinite repeats in the Euclidean plane impossible to represent explicitly [1]. However, the purposes of these illustrations are to demonstrate how to create symmetric patterns and to provide an artistic application of symmetry, rather than to epitomize perfect mathematical accuracy.

3. Illustrations of Border and Wallpaper Patterns

I drew the illustrations below with a combination of ink and gouache (opaque watercolor) paint on bleed-proof paper. The sizes of the illustrations range from four inches square to rectangles that are six inches by eight and a half inches. Each drawing begins with a grid that is either drawn in pencil or ink. Then, I used black micron pens in various sizes to draw the designs in freehand, and in the case of *Symmetry 1*, blue and green gouache colors the design. I use gouache and ink because I can create artwork with a high amount of detail.

3.1 Border Patterns. The grid used in the border patterns is a simple modification of a square grid. This grid is visible underlying the illustrations in Figures 1 and 2. All seven of the border patterns are shown in both figures. Labels are between the two images. I began drawing each pattern in Figure 1 with a simple design that was repeated across the pattern. For example, *m1* starts with a row of smiles, *1g,*and *mg* both started with a row of alternating dots, and *mm* started with a row of circles, all of which are easily visible down the center. Then, each simple pattern was elaborated in two ways, one to the left and one to the right. The inspiration for using recognizable forms came from M. C. Escher. Images include cats, flowers, waves, fish, boats, mouths saying, "CHIC," butterflies, footprints, people, turtles, and ladybugs. In Figure 2, the patterns start simple on the left, become more detailed, and then become simple again on the right.



11
12
m1
1m
1g
mg
mm

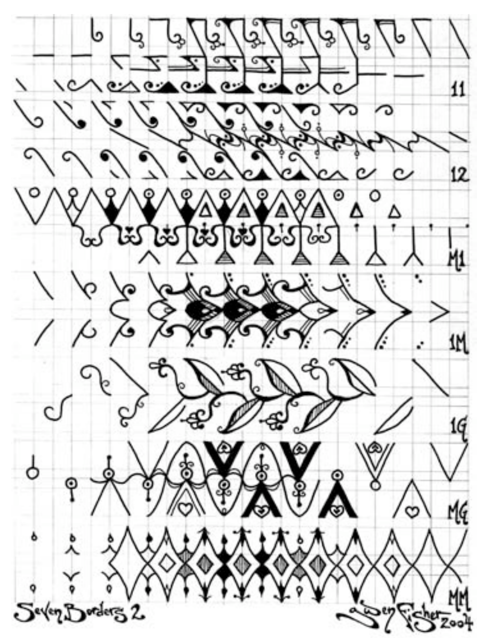


Figure 1: *Seven Borders 1*

Figure 2: *Seven Borders 2*

I began *Seven Borders 5* (Figure 3) with the same grid in Figures 1 and 2. I lightly drew the grid in pencil, which was erased after the drawing was completed in black and red ink. The red ink appears gray in the image below. All seven of the border patterns are visible in *Seven Borders 5*. Labels are on the right of the image. Each pattern began with a simple design that was repeated partway across the pattern. Various patterned marks were added, starting and stopping to keep the density of ink relatively constant across the pattern. The purpose of this illustration is to show many different examples of each of the seven border patterns, all in one image. Notice that several of the design elements are similar in Figures 2 and 3, especially in *11* and *1m*.

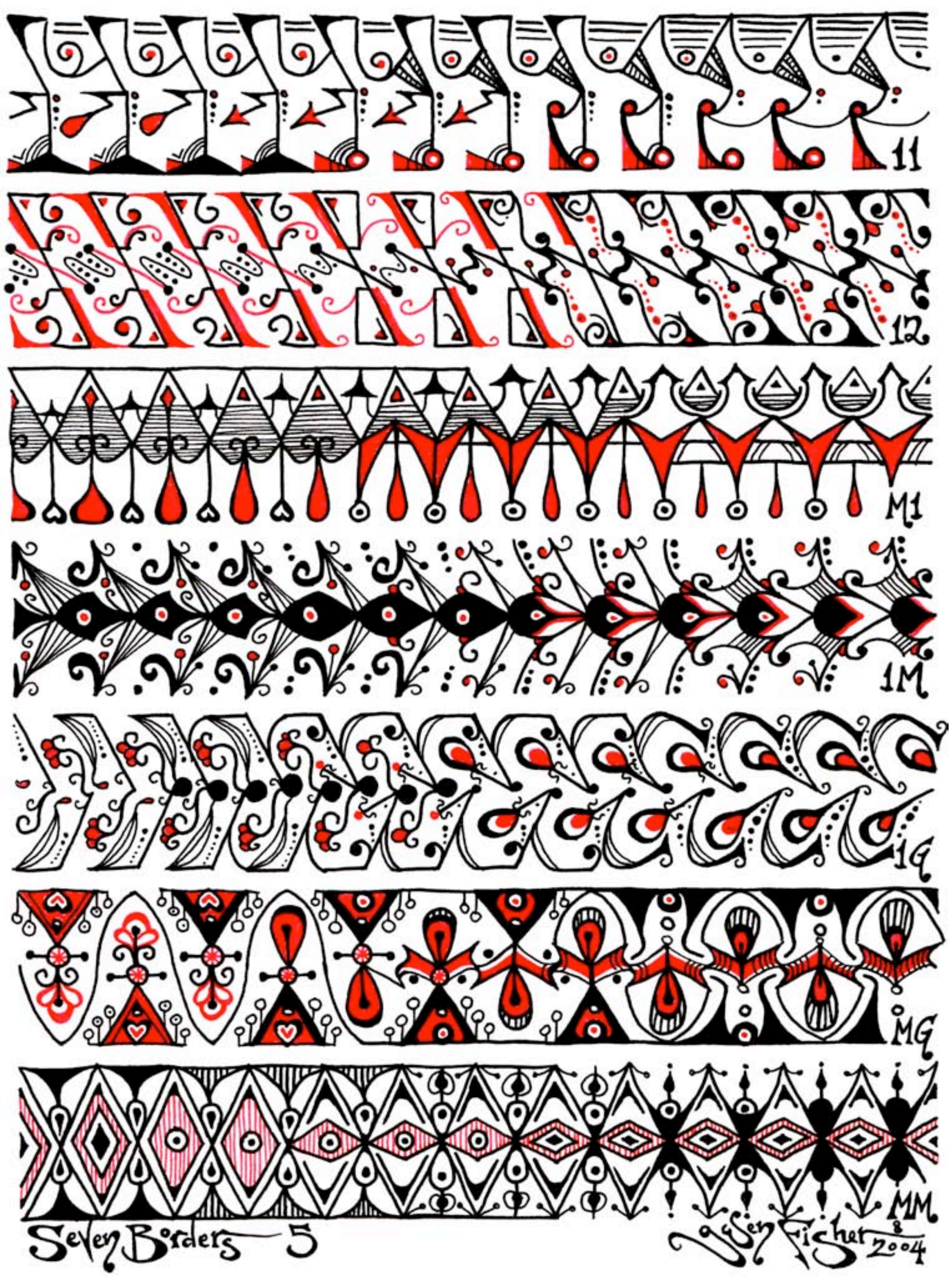


Figure 3: *Seven Borders 5*

3.2 Wallpaper Patterns. I began *Twelve of the Seventeen* (Figure 4) with 15 by 15 square grids drawn in red ink. For each pattern, the illustration shows the placement of the underlying design elements on the top right, and the design gets progressively more complex towards the bottom left. Labels are on the bottom-right of each pattern. *Twelve of the Seventeen* shows representations of 12 of the 17 wallpaper symmetry groups, including all of the wallpaper patterns that can be easily drawn on a square grid, with *p4m* having the same symmetry group as the grid itself. The rotational symmetries found in these patterns include $\boldsymbol{C}_2$ and $\boldsymbol{C}_4$. In fact, $\boldsymbol{C}_4$ symmetry is found only in the three patterns with a "4" in their labels, namely *p4*, *p4g*, and *p4m*. However, $\boldsymbol{D}_4$ symmetry is only found in *p4m*. Since the remaining nine patterns have limited rotational symmetry, these nine patterns can also be easily drawn on an isometric grid.

The *p1*, *pm*, and *cm* patterns are all good for drawing patterns with row after row of objects (such as houses), and the *pg* pattern is particularly nice for vine designs. These four patterns are all of the wallpaper patterns with only trivial rotations. I found *pgg* to be the most challenging of all of the 17 wallpaper patterns to draw. The reason that the *pgg* pattern is difficult may be because it has a great deal of structure, and yet no reflections.

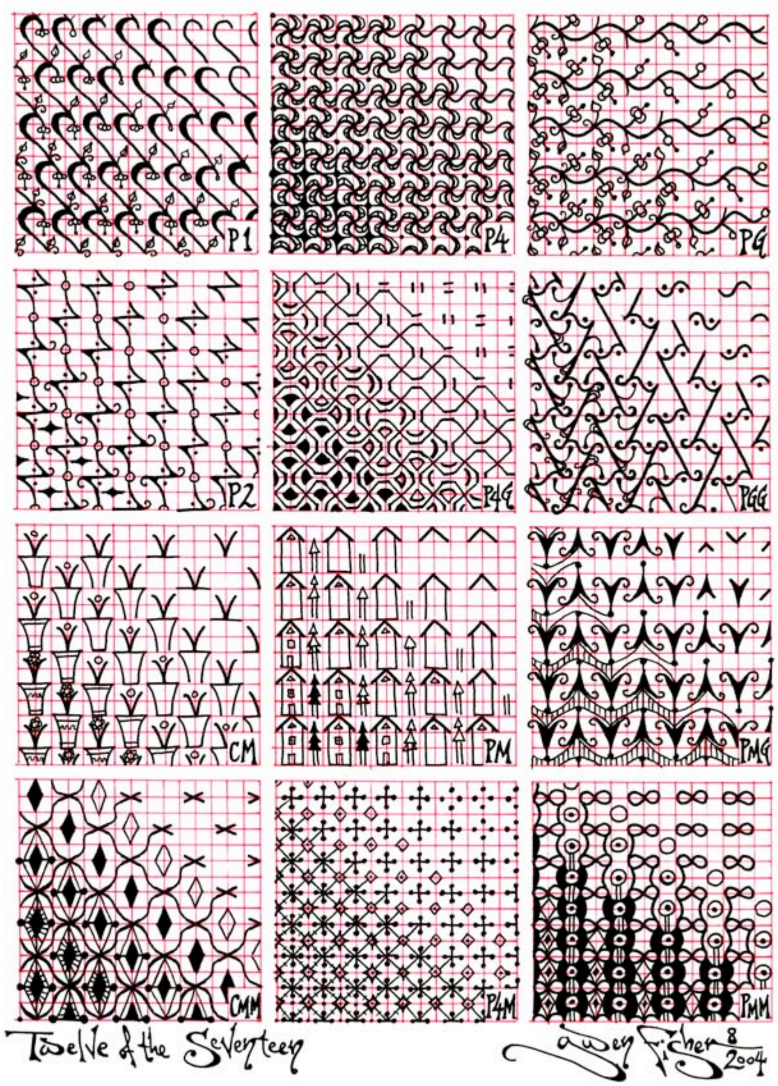


Figure 4: *Twelve of the Seventeen*

Figures 5, 6, 7 and 8 all began with an isometric (i.e., equilateral triangle) grid. Figure 5 shows how to draw the wallpaper pattern with the most symmetry, called *p6m*. This image has three distinct compositions: one on the left side, a second on the top right, and the third on the bottom right. To begin each composition in Figure 5, I first located the $\boldsymbol{D_6}$ centers on vertices of the grid and equally spaced them across the grid. On the left are flowers with six petals; each flower has $\boldsymbol{D_6}$ symmetry. On the top right are dots on each vertex, and on the bottom right are circles centered at each vertex. In each case, the dihedral symmetry is maintained as the patterns increase in complexity.

Notice that *p6m* has the same symmetries of the original triangular grid in the same way that *p4m* has the same symmetries of the square grid. In the compositions on the right side of Figure 5, all of the symmetries of the grid are maintained. In contrast, on the left side of Figure 5, the pattern is scaled by a factor of two, so that the lines of reflection and the centers of rotation are two triangular units apart. To identify this scaling, focus on the side length of the triangles rather than on the area inside the triangles. By scaling the pattern to two triangles instead of one, I was left with more space to embellish the pattern. As long as this type of scaling is consistent in all directions, it does not change the symmetry classification of the pattern as a whole.

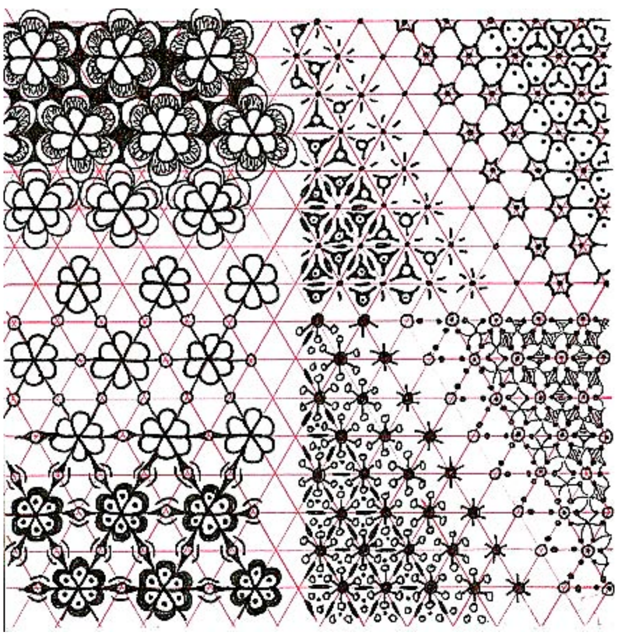


Figure 5: *Drawing* p6m *Wallpaper*

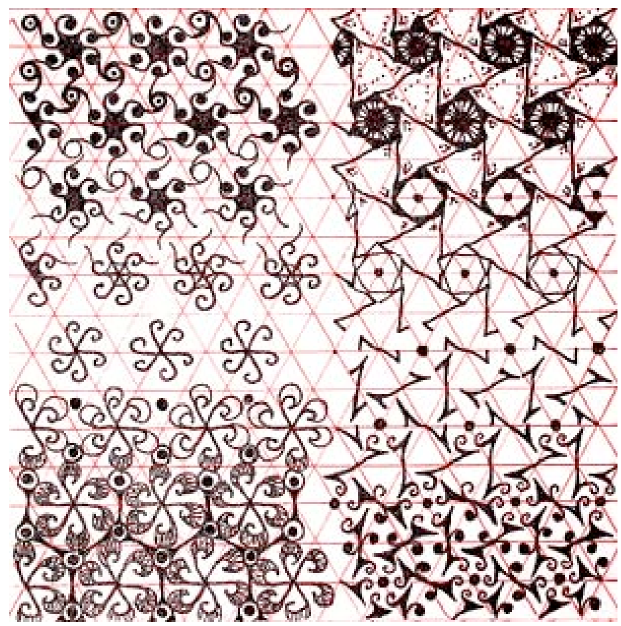


Figure 6: *Drawing* p6 *Wallpaper*

Figure 6 shows what happens when we destroy all of the reflections in patterns like those in Figure 5. Since the mirror reflections in *p6m* are eliminated, and the $\boldsymbol{C_6}$ centers remain, the patterns in Figure 6 are called *p6*. I began this drawing halfway down grid. First, I located the $\boldsymbol{C_6}$ centers on vertices of the grid and equally spaced them across the grid. As in the left of Figure 5, the patterns in Figure 6 are scaled so that the centers of rotation are two units apart. On the left are spiral pinwheels with $\boldsymbol{C_6}$ symmetry. On the right are "Z" shapes that cycle around the $\boldsymbol{C_6}$ centers. Moving up the right, these "Z" shapes connect to form the large tilted triangles. Moving down, I added spirals to the tips and centers of the "Z" shapes.

Figure 7 exhibits two compositions of *p3*. Notice that *p3* has $\boldsymbol{C}_3$ centers and no $\boldsymbol{D}_3$ centers. The pattern on the left is scaled so that the centers of rotation are two triangular units apart, whereas the pattern on the right is not scaled. In both cases, the *p3* patterns display pinwheels centered on the vertices of the grid as well as a two other types of pinwheels centered in the middle of some of the triangles on the grid. For example, starting at the bottom left of Figure 7, a pinwheel with three spiraled prongs is centered on the vertices, and dots sit on the centers of some of the triangles. In drawing the design, I mostly paid attention to keeping symmetric one type of $\boldsymbol{C}_3$ center. The other two types of $\boldsymbol{C}_3$ centers emerged, to some extent, on their own.

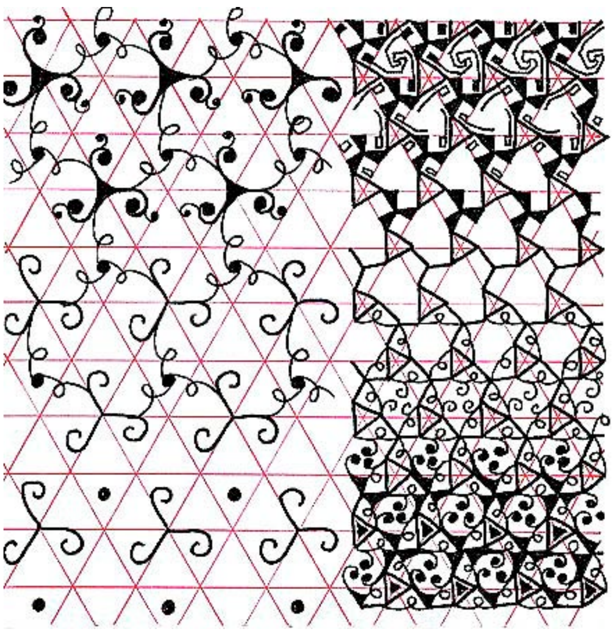


Figure 7: *Drawing* p3 *Wallpaper*

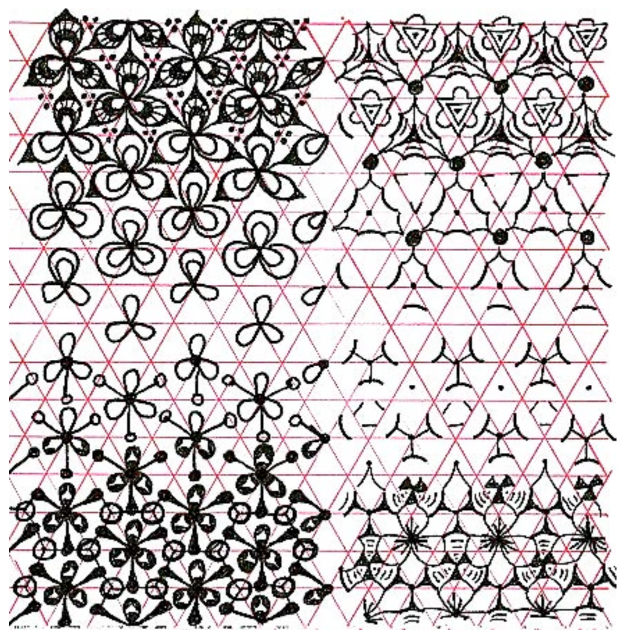


Figure 8: *Drawing* p31m *and* p3m1 *Wallpaper*

We can add reflections to a pattern with $\boldsymbol{C}_3$ centers of symmetry in two different ways. These two patterns are visible in Figure 8, which exhibits *p31m* on the left and *p3m1* on the right. Notice that *p31m* displays left- and right-handed pinwheels centered in the middle of the clusters of three dots in the top left. In contrast, *p3m1* does not have these pinwheels. More formally, we say *p3m1* has centers of $\boldsymbol{C}_3$ symmetry that are not centers of $\boldsymbol{D}_3$ symmetry. The key to drawing *p31m* and *p3m1* is starting with $\boldsymbol{D}_3$ symmetric doodles on the appropriate vertices of the isometric grid. In the *p3m1* pattern (on the right side of Figure 8) the centers of $\boldsymbol{D}_3$ symmetry lie on the vertices with the same type of spacing used in Figures 5, 6, and 7. Here, I used a scaling factor of two. In contrast, the centers of $\boldsymbol{D}_3$ symmetry in *p31m* (see the flowers with three petals in Figure 8, left and center) are staggered along each row of the grid in a way that is different from the centers in Figures 5, 6, and 7. If you view two adjacent triangles as a diamond, then adjacent $\boldsymbol{D}_3$ centers in *p31m* lie on the acute vertices of those diamonds.

The last illustration in this paper shows how an isometric grid can be used to draw the remaining five wallpaper patterns not found in Figure 4. I began *Symmetry 1* (Figure 9) with the grid lightly drawn in pencil, which I erased after the drawing was completed in ink and gouache. Although the isometric grid supports some of the same symmetries as the square grid, this illustration does not incorporate all of the possible symmetries. Instead, *Symmetry 1* includes the five symmetry groups that are subgroups of the symmetries of the isometric grid but not of the square grid. In particular, the top left patterns show *p6* with its $\boldsymbol{D}_6$ centers of symmetry. Moving counterclockwise, the $\boldsymbol{D}_6$ centers transition into $\boldsymbol{D}_3$ centers, showing *p3m1* in the middle left. The *p3m1* pattern continues to the bottom left corner. Continuing around, the pattern changes to *p31m* in the bottom center. The *p31m* pattern continues around the bottom right corner. A third the way up the right side, some of the reflections are destroyed, making *p3*. The pattern changes again in the top right to *p6*. The *p6* pattern is also visible in the left-center of the illustration, highlighted by the pinwheels with six points.

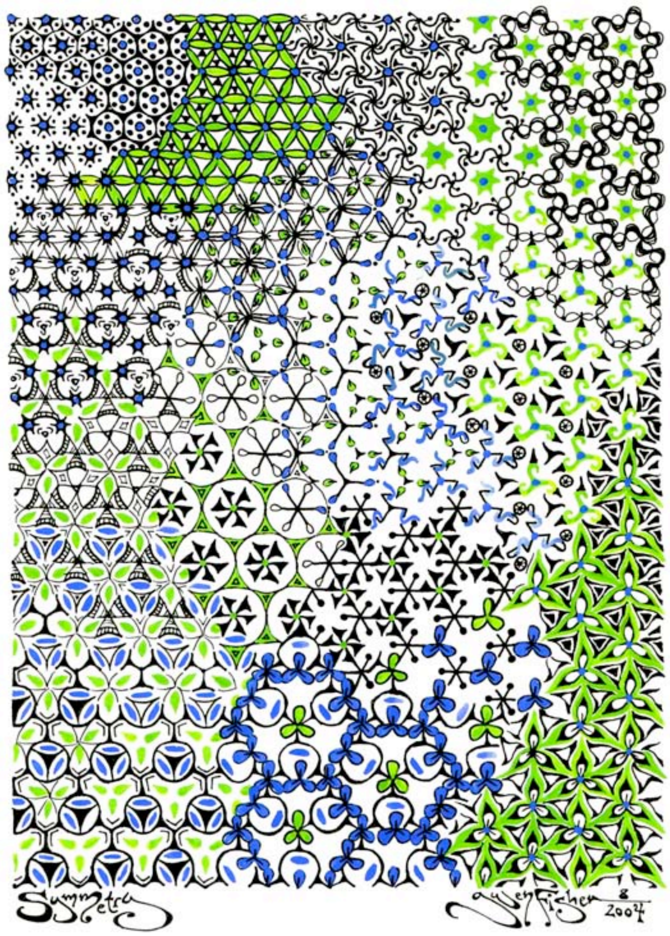


Figure 9: *Symmetry 1*

I have found that illustrating representations of all 24 of the border and wallpaper groups has helped me to better understand the structure of these groups. The use of square and isometric grids provides a scaffold for organizing my doodles into symmetric patterns. I hope that these illustrations will provide a useful foundation for others to design their own sets of symmetric patterns.

4. References

[1] Escher, M. C., *Escher on Escher: Exploring the Infinite*, Harry N. Abrams, Inc, New York, 1989.
[2] Grünbaum, Branko & Shephard, G. C., *Tilings and Patterns,* W. H. Freenman & Co., New York, 1986.
[3] Stahl, Saul, *Geometry from Euclid to Knots*, Pearson Education/Prentice Hall, 2003.