BRIDGES
Mathematical Connections
in Art, Music, and Science

The Mathematics Of Jazz

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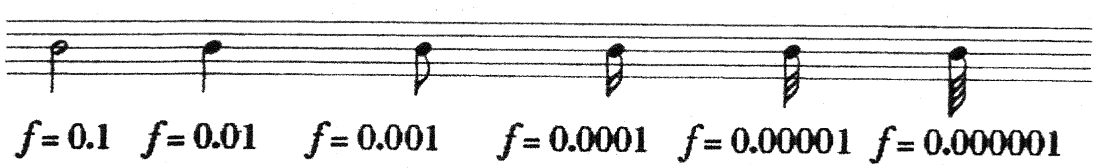
Abstract

There is a sense in which all music has underlying mathematics. There is also, however, a widespread perception that this mathematics is associated only with symphonic music, not with free-flowing, improvisational music such as jazz. It is our premise here that jazz has its own mathematics, which is just as fascinating as the mathematics of a Bach fugue. This underlies such musical fundamentals as the way in which note durations are expressed, the number of bars in a passage, and the actual notes, chords, and key signatures used. The differences between jazz and rock in these respects are also interesting; and jazz musicians have even been known to play sly mathematical games.

1. Durations Of Notes In Symphonic Music

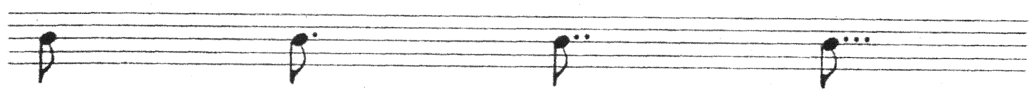
We first take up the ways in which the duration of a note is expressed. In symphonic music there are several conventions for this; and, mathematically, the most important of these is based on binary fractions. Let t be the duration of a whole note in a specific piece of music. Then, in the simplest case, every other note in that piece has (absolute) duration $t{\cdot}f$ where f may be expressed in the binary number system, well known in computer science, as follows:



In the decimal number system, f is $\frac{1}{2}$ (half note), $\frac{1}{4}$ (quarter note), $\frac{1}{8}$ (eighth note), $\frac{1}{16}$ (sixteenth note), $\frac{1}{32}$ (thirty-second note), or $\frac{1}{64}$ (sixty-fourth note).

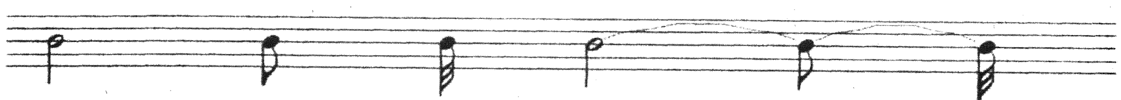
The specification of t here is made by the metronome marking, of the form N = B where N is one of the musical notes above (usually a quarter note) and B is an integer, called the number of beats per minute. Here t, expressed in fractions of a minute, is $1/f\mathrm{B}$ where f is associated with N as above. For example, if N is a quarter note and B is 120, there are 120 beats per minute. Each beat is therefore 1/120 of a minute (or $\frac{1}{2}$ of a second); a whole note is four beats, or 4/120 of a minute, and here this is $1/(\frac{1}{4}{\cdot}120)$ since f, in decimal, is $\frac{1}{4}$. All of the mathematics of durations is independent of t, and indeed, in what follows, we will refer to f, rather than $t{\cdot}f$, as the (relative) duration.

One or more additional one-bits may be added to the binary representations above by the use of dotted notes, with every such one-bit being represented by a dot. Thus, for eighth notes, we have:



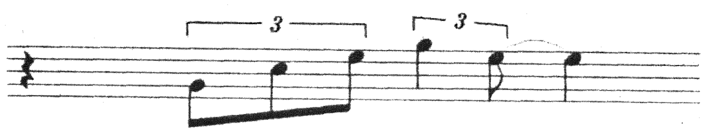
$f = 0.001$ $f = 0.0011$ $f = 0.00111$ $f = 0.001111$

Here these appear more elegantly expressed in the binary than in the decimal, number system, where the relative durations, as above, would be $\frac{1}{8}$, $\frac{3}{16}$, $\frac{7}{32}$, and $\frac{15}{64}$ respectively. All other durations representable as terminating binary fractions may be specified by using ties, which add the tied durations. Thus, for $f = 0.10101$, or $\frac{21}{32}$ in decimal, we may write



$f = 0.1$ $f = 0.001$ $f = 0.00001$ $f = 0.1 + 0.001 + 0.00001 = 0.10101$

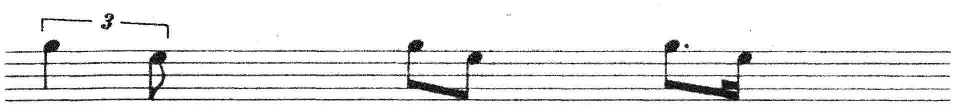
Fractional durations with non-terminating binary representations may be expressed by means of triplets and their generalizations, although here the scheme is not completely general. Let the duration of a note be $t{\cdot}f$ where f is a fraction i/j. If j is a power of two, we have the case above; more generally, $j = k{\cdot}2^d$, where k is odd. In the simplest case, i is 1 and k is 3, and the note is represented as part of a triplet. However, this triplet must be completed, meaning that there are either three notes, all of duration $1/j$, or one of duration $1/j$ and another one of duration $2/j$ (in either order). The total duration in either case is $3/j = 1/2^d$, and the three notes, taken together, take up the space of one note of that duration. The notes of actual duration $1/j$ or $2/j$ are written as if they had duration $1/2{\cdot}2^d$ or $1/2^d$, respectively, like this:



Here i is 1, j is 12, k is 3, and d is 2, so that 2^d is 4. Each of the first three notes has actual duration $\frac{1}{12}$, but it is written as an eighth note, having duration $\frac{1}{8}$. The next note has actual duration $\frac{1}{6}$, but it is written as a quarter note, having duration $\frac{1}{4}$. Triplets may now be generalized to quintuplets and the like, although the scheme is still not completely general.

2. Durations Of Notes In Jazz And The Swing Conventions

All this is different in jazz, where every musical bar may be either **straight** or **swing**. If it is straight, the durations of the notes in the bar are as they are in symphonic music. If it is swing, these durations are modified, and there are two written conventions for this. Both of these are concerned with the specific case of what, in symphonic music, would be a triplet consisting of a quarter note followed by an eighth note. These are written without triplets, as either two eighth notes or a dotted eighth note followed by a sixteenth note, depending on which convention is used, like this:



(as actually played) Convention 1 Convention 2

The reason for the conventions is that it would be tedious to write out all the triplets in a typical jazz piece, which would contain hundreds of them. Both conventions, although not necessarily the mathematics behind them, are well known to all jazz musicians.

There have been occasional attempts to avoid the swing conventions and notate jazz according to symphonic musical notation. Usually these involve the use of either 6/8 or 12/8 time. For example, the prologue to Leonard Bernstein's music for *West Side Story* is in 6/8 time. A single bar in 4/4 time becomes two bars in 6/8 time, with quarter notes in 4/4 time being replaced by dotted quarter notes in 6/8 time. A triplet consisting of a quarter note followed by an eighth note, in 4/4 time, becomes an ordinary quarter note followed by an eighth note in 6/8 time.

3. The Number Of Bars In A Passage

Anyone who has studied jazz is aware of the so-called "twelve-bar blues" and its chord progression (C6-F7-C6-C7-F7-F7-C6-C6-G7-F7-C6-C6, if the piece is in C major). Here C6 refers to the chord C-E-G-A (see section 6 below). The importance of the twelve-bar blues, though, is not in itself, but rather in the contrast between it and the conventional divisions of symphonic music into bars. Here the basic divisions of a given passage are always powers of two: four bars, eight bars, 16 bars, 32 bars; and this holds regardless of the time signature.

4. Jazz Notes

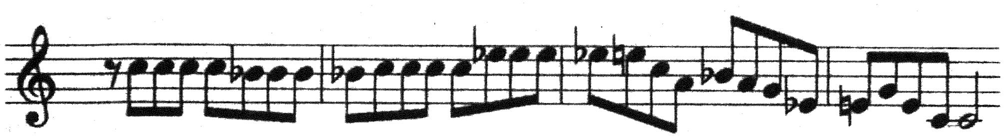
With respect to notes, what distinguishes jazz from symphonic music is the blue notes, which have a mathematics of their own. In the key of C, the blue notes are Bb and Eb; in any other key, the blue notes are the result of transposing Bb and Eb from the key of C into that other key. Mathematically, transposition means multiplying the frequency of every note in a piece by a constant M. When transposing from key K into key K', where K and K' have respective bottom (tonic) notes T and T', with frequencies ω and ω', then $M = \omega' / \omega$. In the special case of transposing an octave up, we have $\omega' = 2\omega$, and every frequency is multiplied by 2.

Levine [1] explains the presence of blue notes in terms of the seventh, which is typically used in modulations. Thus C7 (that is, C-E-G-Bb) contains the blue note Bb, which is used, both in jazz and in symphonic music, in a modulation into F. Also, F7 (that is, F-A-C-Eb) contains the other blue note, Eb. In symphonic music, if F7 is used, it is normally for a modulation into Bb; but in jazz, the modulation is back to C again. This does not happen in symphonic music, except in imitation of jazz, such as in Gershwin's *Rhapsody in Blue*; and it is not explained by the theory of [1].

We propose here a possible alternative explanation of the blue notes, based on cultural considerations at the time, during the nineteenth century, when this music originated. Our belief is that the blue notes arose from a clash of cultural traditions. The European musical tradition involves years of formal lessons, particularly on the violin. In the African tradition, although people often study their instruments for many years, formal lessons are generally unnecessary. Jazz arose from European instruments being played by African Americans, usually without lessons.

Under these cirumstances, certain instruments became favored, such as the piano and the tenor saxophone. When playing an instrument without lessons, it is very common to play it always, or almost always, in C major. However, the tenor saxophone is a Bb transposing instrument. When it is being played in C major, supposedly, it is actually being played in Bb major. Meanwhile, a piano might be accompanying the tenor saxophone. Since the piano is not a transposing instrument, the piano is now playing in C major while the saxophone is being played in Bb major. This gave rise to the blue notes, because Bb major has a key signature of two flats, which are Bb and Eb, as noted above.

One of the blue notes, namely Eb, also appears in minor chords, in symphonic music. Historically, major keys and minor keys have very often been used to represent happy and sad feelings, respectively. There is therefore a resemblance between a sad symphonic piece played in a minor key and a jazz piece played with blue notes, especially if we remember that "blue" can also mean sad. However, this resemblance is only a vague one, both because of the other blue note (Bb) and the fact that jazz also uses the note E (in the key of C major), giving it both a major and a minor feel at the same time. Here is the theme from George Gershwin's *Rhapsody In Blue*, transposed into C major:



Here we have ten blue notes, five occurrences of Eb and five of Bb. On two occasions, the blue note Eb, which is part of the minor chord C-Eb-G, is immediately followed by E, which is part of the major chord C-E-G. This is ubiquitous in jazz, although there are other musical traditions which use both major and minor in the same piece. However, only in jazz does the switch take place within a single measure (and sometimes more than once in the same measure), in this way.

5. Symphonic Chords And Harmonics

Another way in which jazz differs, mathematically, from symphonic music is in its treatment of chords. The basic major chords in symphonic music are based, with one exception, on harmonics, a subject whose mathematics we now review.

When a violin string of length L vibrates at a frequency ω, it can also be made to vibrate in two sections, each of length L/2, by touching it lightly with a finger at the exact center of its length, while using the bow. The resulting frequency is 2ω, and this can be generalized. Thus by touching the string at one-third of the distance from one end to the other, it can be made to vibrate in three sections, each of length L/3, with frequency 3ω. Skilled violinists can produce frequencies of 4ω, 5ω, and 6ω in this way. All such frequencies, of the form $n\omega$ for small integer values of n, are called harmonics of the base frequency ω. Musicians refer to the note with frequency $n\omega$ as the $(n-1)$st harmonic; thus the note with frequency 2ω is the first harmonic, and so on.

The same thing happens in a brass instrument, except that now it is a column of air, rather than a string, which is vibrating. Each setting of the valves of a horn, trumpet, or tuba corresponds to a base frequency ω and harmonics 2ω, 3ω, 4ω, 5ω, 6ω, and 8ω (7ω sounds out of tune and is never played). On a trumpet or a tuba, ω itself is never played, but occasionally one finds 9ω or 10ω. On a (French) horn, one occasionally finds 12ω, or ω itself.

On a trombone, there are seven positions of the slide, from first position (with a corresponding Bb base harmonic) through second and third, with base A and Ab, and down to seventh, with base E. The same harmonics that can be played on a trumpet can also be played on a trombone. This time 7ω also works, although normally only for F# and G above middle C, because the slide position can be adjusted to compensate for the note sounding out of tune. Trombone music does not use ω itself, but trombone players can play it (in first position, anyway) and refer to it as a pedal tone.

In symphonic music, the tonic major chord consists of the third, fourth, and fifth harmonics (that is, frequencies 4ω, 5ω, and 6ω as above) corresponding to some basic ω. Tonic major chords in different keys correspond to different values of ω. The subdominant major chord uses the second, third, and fourth harmonics of $\omega' = 4\omega/3$; in particular, the second harmonic of that, namely $(4\omega/3)\cdot 3 = 4\omega$, is also the third harmonic of ω. The dominant major chord uses the second, third, and fourth harmonics of $\omega'' = 3\omega/2$; in particular, the third harmonic of that, namely $(3\omega/2)\cdot 4 = 6\omega$, is also the fifth harmonic of ω.

The point is that notes which are not harmonics are not played in these chords, with one exception, namely the seventh (as in the dominant seventh). This is sometimes thought of as the sixth harmonic (7ω as described above), but that would be incorrect and out of tune. A trumpet player, for example, could play the sixth harmonic of Bb without depressing any valves, and this would be close to high Ab concert, but it would be flat (that is, lower in pitch than it should be). This is why trumpet players always play that note with the first valve depressed, to get the seventh harmonic of Ab concert, rather than the sixth harmonic of Bb concert.

6. Jazz Chords

There are symphonic chords which are not based entirely on harmonics, such as minor chords, diminished chords, and augmented chords. Jazz chords, however, are much more general than symphonic chords, and are almost never restricted to the harmonics of a base frequency.

The most basic jazz chord is the sixth, which has almost completely supplanted the tonic major chord. Instead of C-E-G, the tonic major chord based on C, one finds C-E-G-A, the major sixth chord based on C; and the A here is not a harmonic of C. Symphonic music has very rarely used this chord, although there are exceptions, such as Peter's theme in Tchaikovsky's *Peter and the Wolf*:

The fifth chord in this measure is C-E-G-A, and the others are variations of it (E-G-A-C, G-A-C-E, and A-C-E-G). Further exceptions occur in the music of Rameau.

The basic mathematical rule of jazz chords is that, for every two notes (call them X and Y), selected arbitrarily from the twelve possible tones (C, Db, D, Eb, E, F, F#, G, Ab, A, Bb, and B), there is at least one major chord whose bottom (tonic) note is X and whose top note is Y. This rule allows for maximum flexibility in the writing of jazz; the composer can choose top and bottom notes arbitrarily and can then always fill in the middle notes to make a recognizable jazz chord, as long as the piece is in a major key.

It suffices to show this for the case in which X is C, and the chord in based on C major, since any other values of X may be obtained from these by transposition. Here are some of the basic jazz chords based on C:

The top notes of these chords include A, Bb, B, Db, D, Eb, F, F#, Ab, and (high) A. This leaves only C, E, and G, which are already part of a C major chord and may be added at the top of that chord without change to the nature of the chord.

Minor chords do not follow the rule. Thus, although the C7+9 chord above (C-E-Bb-Eb) is well accepted in jazz, the opposite (C-Eb-Bb-E) sounds harsh and grating, and is never used. Indeed, there are no C minor chords at all in jazz which have E at the top. However, there are minor chords which are commonly used in jazz and rarely, if at all, used in symphonic music. An example is Am+7 (A-C-E-G#).

Another difference between jazz and symphonic music is concerned with final chords. The last chord of an entire piece in symphonic music is almost always either a tonic chord or a bare tonic, with all instruments playing the same note. This rule is so universal that many people find it highly unsettling to hear it broken, as in the song, *I Wonder As I Wander*, which ends on the subdominant:

In jazz, as in symphonic music, the bare tonic will sometimes be used to end a piece, such as *In The Mood*, which is in Ab major and which ends on a bare low Ab. However, it is more common in jazz to end a piece on one of the jazz chords, although still based on the tonic.

7. Symphonic And Jazz Keys And Instrumentation

The sharp keys are favored in symphonic music; the flat keys are favored in jazz (as they are, also, in the concert band). This is well known to arrangers who are choosing a key for a singer; if A is the key in which the singer feels most comfortable, the jazz arranger will try Ab or Bb instead. It is so pronounced that even when musicians tune up, symphonic musicians always tune up to an A (three sharps), while jazz musicians almost always tune up to a Bb (two flats).

Part of the reason for this is open tuning of the string and brass instruments. Every string of a violin, viola, cello, or string bass is tuned to either a sharp key (G, D, A, or E) or to C. This note results when the open string is played (that is, without fingering). By contrast, open tuning of a brass instrument is to a flat key (F or Bb). This includes the horn, trumpet, or tuba played without depressing any valves, or the trombone played in first position.

Wind instruments played in jazz are likewise oriented toward the flat keys. Alto and baritone saxophones are Eb transposing instruments; clarinets and tenor saxophones, like trumpets, are Bb transposing instruments. Wind instruments played in the symphony, however (except for clarinets, which are found in both places) are oriented toward C major. This includes the flute, piccolo, oboe, and bassoon. The English horn, a lower version of the oboe, is actually a G transposing instrument.

This is not the full story, because trumpets are also found in both places, and French horns are symphonic, not jazz, instruments. Also, one occasionally finds jazz musicians who play the violin, such as Stephane Grappelli. However, violins and other string instruments dominate the symphony, which was originally based on these, the other instruments being added later, such as trombones.

8. The Mathematics Of Rock

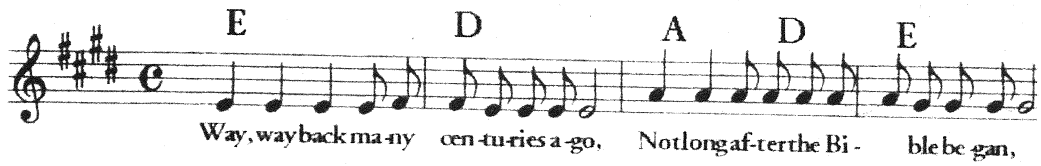
Rock music sounds very different from jazz; and, mathematically, the reason is that rock music has abandoned triplets entirely. In rock music, there are still eight, or sometimes sixteen, beats to the measure, but these are straight, in the above sense. Only in early rock, such as Bill Haley's *Rock Around The Clock* or the Beatles' *All My Lovin'*, do we find triplets.

Also, rock music uses symphonic keys (that is, sharp keys) rather than the flat keys of jazz. This is because a guitar, like a violin, has open tuning on the sharp-key notes: G, D, A, E, and B. Guitars dominate rock music in the same way that violins dominate symphonic music.

Rock music is also different chordally, particularly in its use of what might be called, for lack of a better name, the double subdominant. When the key of C major is the tonic, G major is the dominant; when G major is the tonic, D major is the dominant. Therefore, when C major is the tonic, D major is the dominant of the dominant, or the "double dominant," also known as the supertonic.

In symphonic music, however, there is no corresponding subdominant of the subdominant. When C major is the tonic, F major is the subdominant; when F major is the tonic, Bb major is the subdominant. Therefore, when C major is the tonic, Bb major is the subdominant of the subdominant, or what we are calling the double subdominant. There are a few examples of this in symphonic music, like this passage, near the end of Dvorak's *New World Symphony*:

Here the tonic is E major, and D major is the double subdominant. This theme permeates rock music; indeed we occasionally find an entire piece based on the double subdominant, such as *Jacob and Sons* from Andrew Lloyd Webber's *Joseph and the Amazing Technicolor Dreamcoat*:



This chord progression of E major, D major, A major, D major, E major is used throughout the piece. It is particularly easy for guitars; and the electric guitar, ubiquitous in rock music, uses the same tunings as the ordinary acoustic guitar.

9. Playing With Mathematics

Finally, jazz is not immune to the kind of playing with mathematics that characterized Bach fugues. Consider, for example, the cycle of keys, namely C-F-Bb-Eb-Ab-Db-Gb (= F#)-B-E-A-D-G-C. This is arranged according to the numbers of flats and sharps in the corresponding key signatures. Thus the key of F major is one flat; the key of Bb major is two flats; and so on up through Gb, which is six flats, but which is also F#, or six sharps. Then the key of B major is five sharps, the key of E major is four sharps; and so on down through G, which is one sharp, and back to C again. Every two adjacent notes in the cycle differ from each other by either a perfect fourth or a perfect fifth.

This cycle is well known to musicians, but it would not seem particularly melodious. Yet there is a jazz piece, *Harlem Nocturne*, in which the cycle is played, and a downward chromatic scale is played at the same time, like this:

Omitting the "pickup note" F at the beginning, and the final A, we see from the top notes of each of the chords above that this starts on G, near the end of the cycle. It then proceeds with the entire cycle once around, from C to C and then starting the cycle a second time, ending on Bb. The simultaneous chromatic scale in the bottom notes of each of the chords above starts on B and proceeds downwards, B-Bb-A-Ab and so on down to another Ab. The resulting chords are portions of G7, C7, F7, Bb7, and so on around the cycle.

References

[1] M. Levine, *The Jazz Theory Book*, Petaluma, CA: Sher Music, 1995.