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Constructing and Classifying Designs of al-Andalus

B. Lynn Bodner
Mathematics Department
Cedar Avenue
Monmouth University
West Long Branch, New Jersey, 07764, USA
E-mail: bodner@monmouth.edu

Abstract

This paper will illustrate the mathematical underpinnings of some of the most well-known and best-preserved artistic designs of Islamic Spain from the 13^{th} – 15^{th} centuries. The art objects incorporating these elaborate designs include geometric mosaics, decorative ceramic work, carvings in ivory and wood, and metal work. Slides of the objects will be shown and their idealized geometrical structure then explored with the aid of computer-drawn compass and straightedge constructions, using the *Geometer's Sketchpad* software. The crystallographic symmetry group, and the affect of colorings on this, may also be discussed for each design.

Introduction

One characteristic of Islamic art is the tiling of flat surfaces with overall geometric designs. The underlying principle and structure of these designs may be a framework of identical *repeat units*, or *motifs*, that regularly recur to form a geometrical grid or a regular division of the plane. To create the repeat unit, the Islamic artist could use the geometer's tools of straightedge and compass to draw circles, and within the circles, polygons of all shapes and sizes. Equilateral triangles, squares, pentagons, hexagons, octagons, star polygons and other shapes could be constructed. These polygons could then be further divided to form a seemingly unending variety of geometric patterns.

Although the variations among the designs may appear to be unlimited, in fact, they all may be described mathematically as belonging to only a finite number of possible classes, based on the transformational symmetries they permit. These symmetries include *rotations* about a point (called a center of rotation) through a given angle, *translations* in a given direction through a given distance, mirror *reflections* in a line (called a mirror line or mirror of reflection), and *glide reflections* which combine translations through a given distance and parallel to a line, and then a reflection in the line. To tile a flat plane, there are only 17 different symmetry groups (called crystallographic symmetry groups) possible, involving the various combinations of rotations, translations, reflections, and glide reflections. It should be noted that the crystallographic restriction only allows one-fold (the identity transformation), two-fold, three-fold, four-fold or six-fold rotations [1]. When considering the color symmetry of designs, additional classes of the two-dimensional symmetry groups are possible [2].

In this paper, a few examples of art objects from artisans working in Islamic Spain during the 13^{th} through the 15^{th} centuries will be illustrated. These will include geometric mosaics, decorative ceramic work, carvings in ivory and wood, and metal work. The designs will be analyzed for their symmetry elements and classified to determine the underlying geometrical structure of each motif. The designs will then be recreated in a manner described by El-Said and Parman [3] using the *Geometer's Sketchpad* software [4]. The affect of colorings of the patterns on the crystallographic group classification for some of the designs will also be discussed.

Two Designs, Classified as *p*4*m* (or *p*4 When Considering Interlacing)

The first pattern, involving eight-pointed stars and cruciforms, is a very popular one that may be found widely throughout the Islamic world from Spain to Central Asia. It is incorporated into the design on the handle of a sword belonging to the last Nasrid ruler, Muhammad XII, also known as Boabdil (Figure 1.). The sword, captured in the battle of Lucena in 1483, is in the collection of the Museo del Ejercito in Madrid [5]. An idealized rendition of this design, without the interlacing, is given in Figure 2.

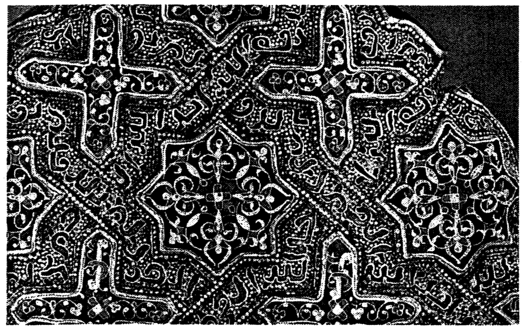

Figure 1. Detail of handle of Boabdil's sword

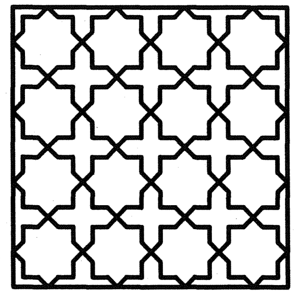

Figure 2.

The design (without the interlacing) appears highly symmetric with at least a four-fold rotation permissible about the center of the eight-pointed star. Upon further examination, one may notice that two mirror lines may be drawn through the center of the axes of the cruciform and two additional mirror lines may also be drawn at 45 degree angles to these axes. Thus with mirror lines in four directions, it appears that one may obtain the basic motif (Figure 3.) from the repeated construction of squares.

Specifically, to create this motif, one may start by constructing a square and connecting the opposite vertices (or corner points) of the square with line segments (diagonals) to find the center of the square. Then one may find the midpoint of each side of the square. Using the center of the square and one of the midpoints, one may then construct an inscribed circle and find the points of intersection between the circle and the diagonals (Figure 4.)

Next, by drawing four line segments connecting the midpoints of the original square a smaller square with sides at a 45 degree angle to the sides of the first square may be formed. Connecting the four remaining points of the circle (that is, the points that are the intersections of the circle and the diagonals) with line segments forms a second smaller square with sides parallel to the sides of the first square (Figure 5.). Notice that these two smaller squares are congruent. By erasing some of the line segments of these two squares, one may form an eight-pointed star polygon (Figure 6.) Erasing additional lines and the circle, the motif inscribed within a square is complete (Figure 3.) and ready to be duplicated.

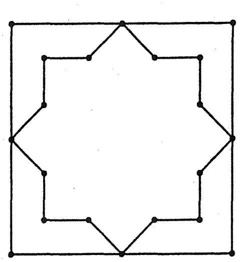

Figure 3.

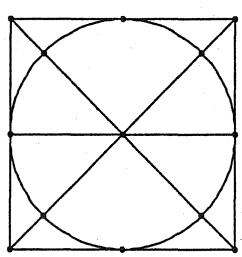

Figure 4.

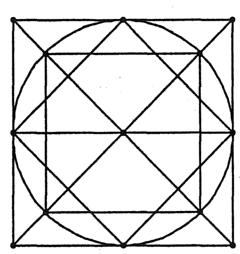

Figure 5.

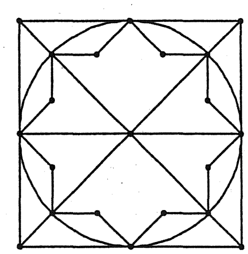

Figure 6.

To form a seemingly unending two-dimensional design with no overlaps or gaps, one may either translate (at right angles to any of the sides of the square) this motif by the dimension of the square, or rotate the motif 90 degrees about a corner vertex of the square, or reflect it across any of the sides of the square. The multitude of different ways of tiling the plane are due to the high symmetry of the idealized pattern, which has a crystallographic group classification of *p*4*m*, since it permits both four-fold rotations with mirror reflections in four directions. When considering the actual pattern found on the sword handle, the classification would be *p*4 since the interlacing design does not allow mirror reflections.

The second design common in Islamic art, consisting of octagons and four-pointed stars, is displayed below on a 14th century cedar writing desk from the Nasrid period (Figure 7.). The desk, in the collection of the Museo Arqueologico Nacional in Madrid, is decorated with marquetry and inlaid ivory [5]. This octagon design may also be observed on the handle of a glazed and painted earthenware Alhambra vase, also from the Nasrid period, late 13th century (Figure 8.), found in the collection of the Galleria Regionale della Sicilia, Palermo [5]. An idealized form of the basic pattern (without the embellishments within the octagons or the interlacing) is given in Figure 9.

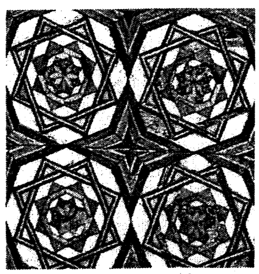

Figure 7. Detail of writing desk

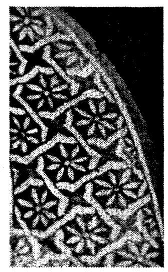

Figure 8. Detail of Alhambra vase

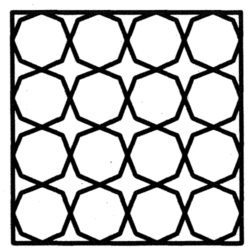

Figure 9.

As with the first idealized design, the crystallographic group classification for this pattern is *p*4*m* since it also permits a four-fold rotation about the center of the octagon and has mirror lines of reflection in four directions. The four mirror lines are the lines drawn through opposite vertices of the square and through the opposite midpoints of the sides of the square.

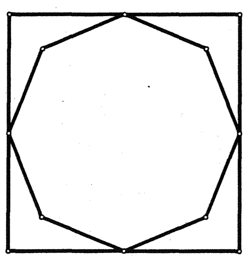

Figure 10.

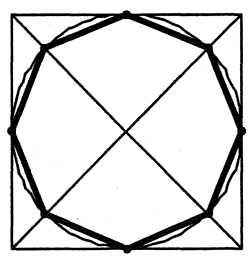

Figure 11.

The basic repeat unit (Figure 10.) may be created by repeating the steps described above to obtain the design in Figure 4. and then connecting the points of intersection between the diagonals and the circle with the midpoints of the sides of the square (Figure 11.). A regular octagon is thus inscribed within the circle. Erasing the circle and all the line segments, except for those forming the octagon, yields the motif in Figure 10. To tessellate the plane, one may either translate (at right angles to any of the sides of the square) this motif by the dimension of the square, or rotate the motif 90 degrees about a corner vertex of the square, or reflect it across any of the sides of the square. When considering the actual pattern found on the Alhambra vase, the classification would be *p*4 since the interlacing design does not allow mirror reflections.

Maple Leaf Design, Classified as *p*4*g* (or *p*4'*g*'*m* When Two-Colored)

A third design, known as a "maple leaf" pattern, may be seen as part of a ceramic wall mural in the Alhambra of Granada, Spain (Figure 12.). An idealized, computer-drawn outline of the pattern is given in Figure 13. Without any coloring, the design is classified as *p*4*g*, since it allows four-fold rotational symmetry and a mirror reflection not on mirror lines that intersect at 45 degrees.

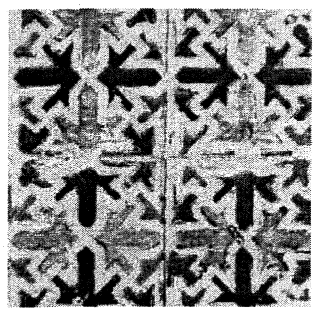

Figure 12. "Maple leaf" mural at the Alhambra

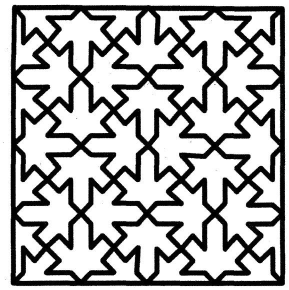

Figure 13.

The construction of this design also begins with the construction of two squares inscribed within a circle at 45 degree angles to each other as was the case for the first design, and then a third smaller square is nested within one of these squares (Figure 14.). Keeping the lines in bold print in Figure 15. and erasing the others, the maple leaf motif becomes evident (Figure 16).

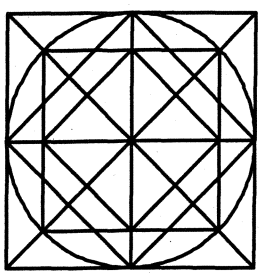

Figure 14.

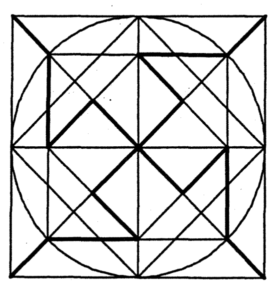

Figure 15.

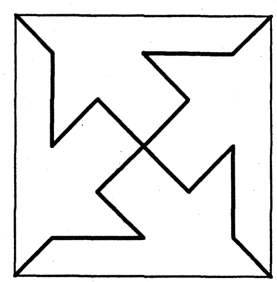

Figure 16.

To form an uncolored tessellation of the plane (Figure 13.), one may reflect the repeat unit found in Figure 16. across any of the sides of the square or rotate it 180 degrees about any of the vertices of the square. A perfect two-coloring of the design (Figure 17.) will change the crystallographic group classification to *p*4'*g*'*m* (which has 2-fold rotational symmetry and mirrors in two directions).

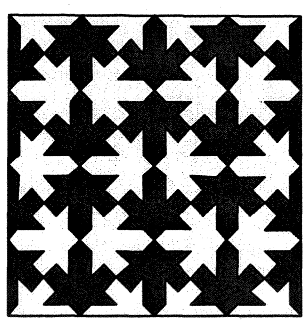

Figure 17.

Hat Design, Classified as *p*4*g* (or *p*4*'g'm* When Two-Colored)

A fourth design, known as a "hat" design, is also found in wall murals at the Alhambra (Figure 18.). An idealized computer-drawn, uncolored rendition of the pattern is given in Figure 19.

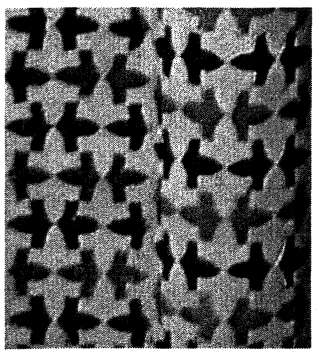

Figure 18. "Hat" design mural at the Alhambra

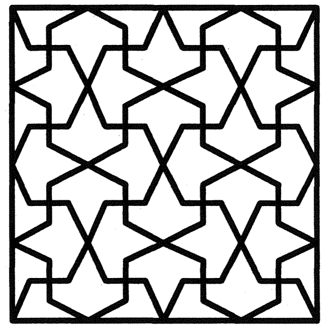

Figure 19.

The hat design may be constructed from a square. The midpoints of the sides are found and connected by line segments to form four smaller squares, each sharing the center point as a common vertex. The midpoints of the four segments interior to the larger square are then found and segments drawn from these four points to the four vertices of the largest square as shown in Figure 20. After erasing a few segments, the uncolored motif is revealed (Figure 21.). A tiling of the hat motif with perfect two-color symmetry is given in Figure 22.

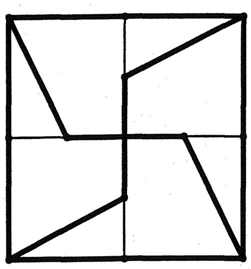

Figure 20.

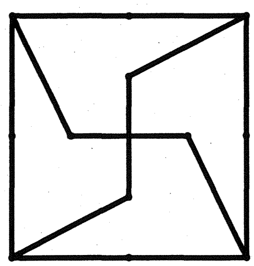

Figure 21.

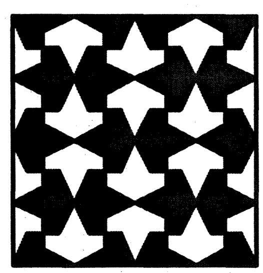

Figure 22.

To form a tessellation of the plane, one may either rotate the motif 180 degrees about any of the vertices of the square or reflect it across any of the sides of the square. If one considers the Alhambra mural as a two-coloring consisting of only white hats and dark hats, the crystallographic group classification for the pattern would be *p*4*'g'm*.

Key Design, Classified as *p*4

A fifth design, known as the "key" design, is another one found in wall murals of the Alhambra (Figure 23.). An idealized computer-drawn rendition of the pattern (without the embellishment within the eight-pointed star) is given in Figure 24. Its eight-pointed star design suggests that the motif may also be constructed from squares (Figures 25. and 26.) in a manner similar to the previous examples. The pattern permits no mirror reflections, but it does allow a four-fold center of rotation at the center of the star. Therefore, the idealized design has a crystallographic group classification of *p*4.

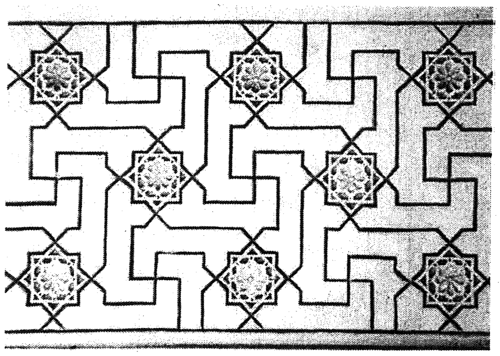

Figure 23. 'Key' motif in mural at Alhambra

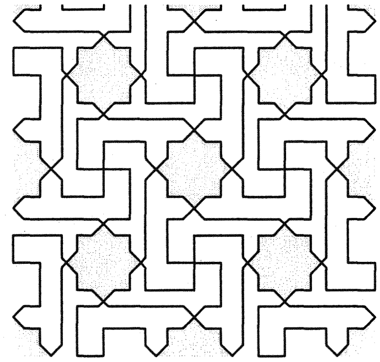

Figure 24.

To tessellate the plane with the motif in Figure 27., one may either translate (at right angles to any of the sides of the square) the motif by the dimension of the square, or rotate the motif 90 degrees about a corner vertex of the original square.

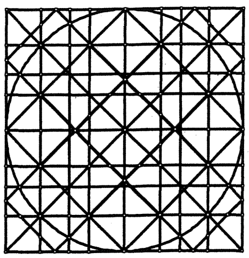

Figure 25.

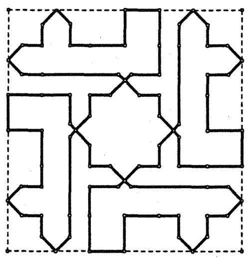

Figure 26.

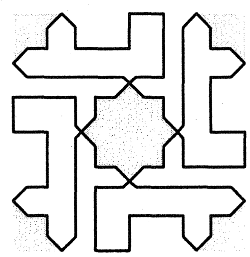

Figure 27.

Bird Design, Classified as *p*6 (or *p*6' When Two-Colored)

Finally, we get to crystallographic groups without a four-fold symmetry of any kind. Designs with three-fold or six-fold symmetries are found much less common in the art of Islamic Spain and so we will only have space enough to discuss one such design. The "bird" design is widely seen throughout al-Andalus. The example given in Figure 28. may be found in a wall mural at the Alhambra. An idealized skeletal copy of the design is given in Figure 29.

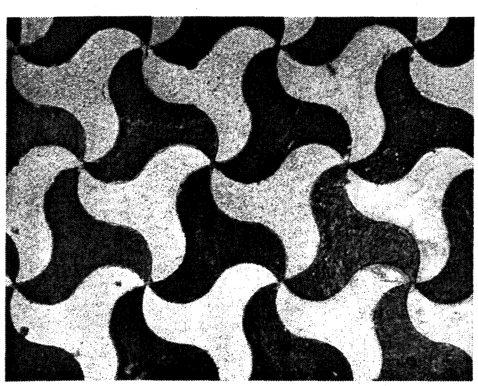

Figure 28. "Bird" motif on mural at the Alhambra

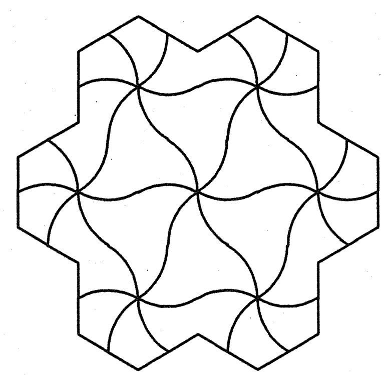

Figure 29.

As one might expect, a motif with six-fold symmetry may be based on the structure of a regular hexagon. To construct a hexagon, one first draws a line segment, an arbitrary point is chosen to be the center of a circle, and a circle is drawn. The points of intersection between the circle and the line are found and then used to construct two more circles, on opposite sides of the circle, using the center of the first circle to determine the radius. The points of intersection of these two circles with the original circle, along with the two points of intersection between the original circle and the line, become the vertices of an inscribed hexagon within the original circle (Figure 30.) Connecting the midpoints of the three non-adjacent sides of the hexagon yields a six-pointed star. To construct an S-shaped curve, which will eventually form one of the three sides of the bird, construct two more circles so that for each circle, one point of the star will be the center, with a radius extending to one of the two adjacent vertices of the star (Figure 31.).

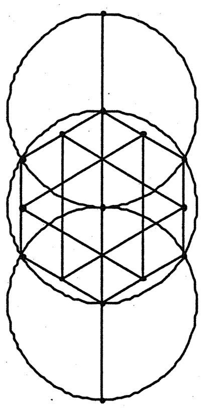

Figure 30.

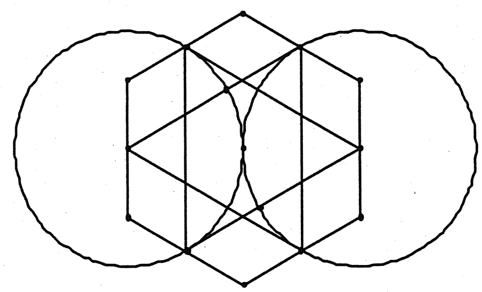

Figure 31.

Repeat this procedure twice to get the two other S-shaped curves, and then erase the arcs of the circle and the six-pointed star (Figure 32.) to obtain the motif in Figure 33.

To tessellate the plane with the uncolored bird motif, one may either translate the motif perpendicular to any side of the hexagon by the distance from two midpoints of opposite sides of the hexagon, or rotate the motif 120 degrees about a corner vertex of the hexagon. Since the design permits no mirror reflections, it is classified as *p6* if uncolored and therefore has six-fold rotational symmetry as shown in Figure 29., or it is classified as *p6'* if it is two-colored as in Figure 34. If one considers the coloring of the Alhambra mural to consist of white birds and dark birds, the design would be also be classified as *p6'* because the white and dark birds alternate colors.

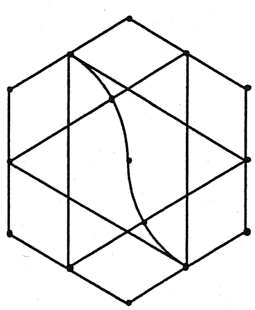

Figure 32.

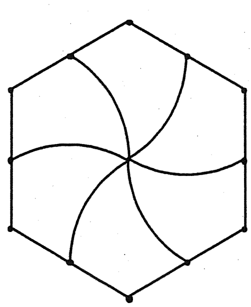

Figure 33.

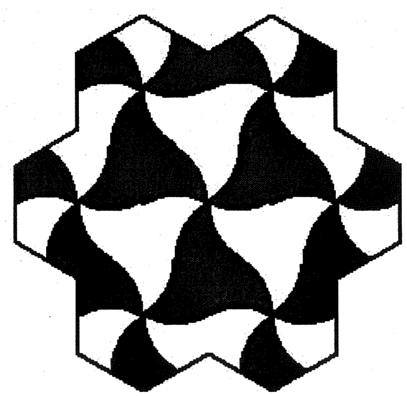

Figure 34.

A further enhancement of the bird design may be found as a wall mural at the Alhambra in the Court of the Myrtles, with alternating six-pointed stars and regular hexagons found within the birds. To achieve this, one may alter the basic motif of Figure 33. by adding line segments as outlined in Figure 35. and then tessellating the plane with the motif given in Figure 36. To obtain the idealized colored rendition provided in Figure 38., classified as *p*3 since it allows three-fold rotational symmetry, tessellate with the motif given in Figure 37.

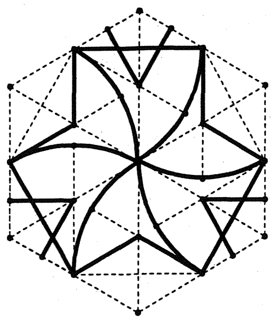

Figure 35.

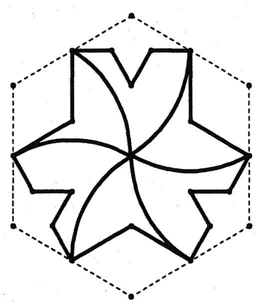

Figure 36.

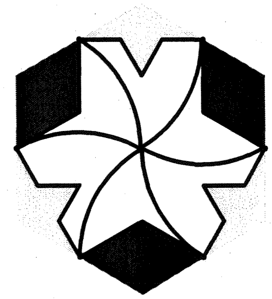

Figure 37.

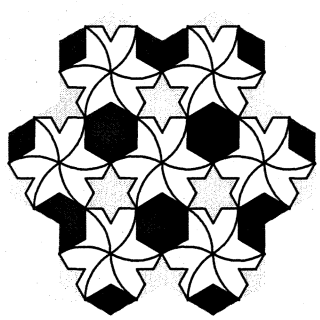

Figure 38.

Conclusions

The extraordinary artistic designs of al-Andalus may be analyzed and classified according to the symmetry transformations the patterns allow. These, in turn, assist our reconstruction of the designs using an electronic version of the geometer's compass and straightedge tools. When one is able to find the hidden structures and relationships between the elements of the designs, the patterns become even more intriguing.

In this paper, only six Islamic designs from al-Andalus were classified and constructed. It is sincerely hoped that the reader will be inspired to continue to look for the mathematics underlying the many elaborate patterns of the art of Islamic Spain.

References

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