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On the Shapes of Water Fountains and Times Tables

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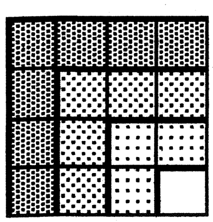
Abstract

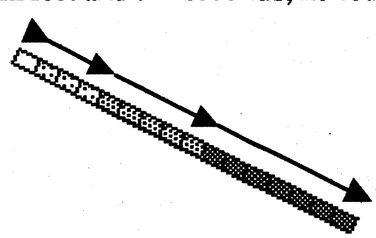
According to the well-known Galileian approximation (near Earth's surface, ignoring air resistance), the trajectory of a freely falling body such as a ball or drop of water is a parabola; but what of the many drops of water springing from a fountain (under equal pressure but at different angles to the horizontal) — what describes its over-all profile? Apparently unrelated, the usual shape of a times table is a semi-square since the commutivity of ab = ba makes the other half of the square redundant; are there different naturally-motivated shapes in which to display such facts of elementary arithmetic? In particular, what if the multiplication is in a finite modular system? Surprisingly, the two questions turn out to be related and lead to Möbius strips in special cases.

1. The Fountain from Florentine Pythagoreanism to Newton's Physics

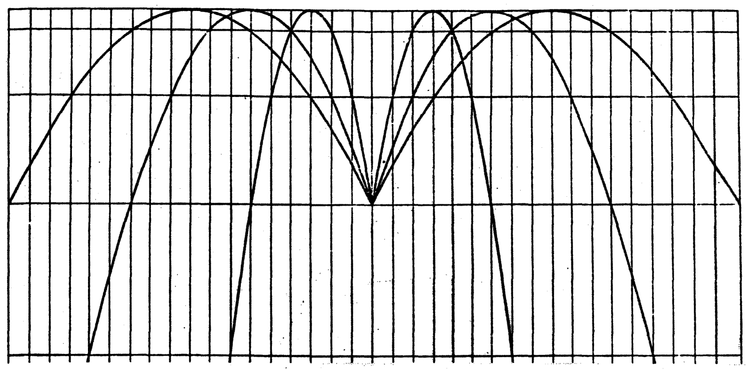
The 16th century Florentine *Camerata* was by intent concerned with reviving or giving rebirth (*renasciamento*) to the culture of classical Greece, yet its results often led down new pathways. Its leader, Vincenzo Galilei, was attempting to revive the practices of Greek drama by use of χορος and ορχηστρα but wound up becoming the composer of some of the first Italian operas, and his lute book of 1584 was one of the first music publications to experiment with equal temperament (i.e. using Stevin's new irrational 12th roots of 2 rather than Zarlino's classical ratios of small whole numbers) [1].

Galileo Galilei, Vincenzo's better-known son, was arguably the reverse: by intent a modern empirical experimentalist who occasionally inadvertently revived some ancient Pythagorean theories, as when he discovered his law of freely-falling bodies. The ancient Greeks had viewed figurate numbers as growing by successive addition of similar regions; in the case of square numbers, these were Γ-shaped Γνωμοι of successive odd-numbered lengths so that $1^2 = 1$, $2^2 = 1+3$, $3^2 = 1+3+5$, $4^2 = 1+3+5+7$, etc., as shown at left below. What Galileo discovered by rolling balls down incline planes was that they would pass through 1, 3, 5, 7, etc. units of space during successive equal units of time, as at right. Knowing their sums yielded squares he formulated $s = -kt^2$, the ball's spatial distance travelled (*situs* s) being proportional to the square of the elapsed time (*tempus* t), the proportionality factor k being dependent upon angle of inclination and choice of measurement units; in the case of a vertical drop with s in feet and t in seconds, he found $k \approx 16$.





When I decided to become a teacher, I did so with the intent that my students and I would feel free to correct one another in real time (like the journal referee process but much faster — "instant karma"). The first such correction I experienced occurred when, as a novice, I uncritically lifted an illustration from a book by the master Hermann von Baravalle (who was on the faculty of the original Waldorf school in Stuttgart in the 1920's, emigrated to the United States in the 30's, and was instrumental in founding most of the original Waldorf schools in this country in the 40's, contributing numerous articles to *The Mathematics Teacher*, and often cited by Martin Gardner in his columns in the *Scientific American*). In its pre-war version, *Das Reich geometrischer Formen* [2], p. 28, the figure consisted of parabolas through a common point drawn by using square grid paper to move first 5,3,1 units up and then 1,3,5,7, etc. units down, while moving a set number of units to left or right, as shown below. The text simply stated that the results "correspond to the laws of free fall" since they obey Galileo's law in their same vertical movement (while remaining unaffected in their various horizontal movements). In its post-war version, *Geometrie als Sprache*



der Formen [3] (the source of his American pupil Col. Beard's *Patterns in Space* [4]), the same drawing appeared as Fig. 271 but the text had been fatefully extended to refer to the collection of such parabolas as describing the trajectories of droplets of water in a fountain (*Springbrunnen*). A (typically back-row) student instantly nailed me by saying "Fountains aren't flat on top!" And indeed, to appear as above, jets at angles departing from the vertical would need to emerge under ever increasing pressures to reach the same maximum altitudes making fountain "flat on top," approaching impossible infinite pressure when emerging in the horizontal direction. Jets emerging from a natural fountain under equal pressure from an ideal spherical nozzle would give a collective profile that is somehow rounded, not flat — but how? What is their true profile shape? The student question was from Dana Williams (now on faculty of the Northridge Waldorf school, Highland Hall), and the physics colleague who helped me answer it was James Huston (then at High Mowing, Wilton, N.H., and subsequently on faculty of Wentworth University in Boston). This presentation is dedicated in joint gratitude to them, and is based on letters which grew out of discussions with them and others, excerpts from which were printed (with permission) in several issues of an informal research journal which I edited quarterly from 1972 to 1982 [5,6,7,8,9].

If we imagine the fountain's source placed at the origin (0,0) of a Cartesian coordinate system with typical jets leaving at angles θ = 0°, 15°, 30°, 45°, ... all at same initial velocity V_o, and the central jet (at θ = 90°) achieving maximum height h, then the trajectories of droplets in its profile plane may be described as loci of points (x,y) satisfying

$$V_{x_0}^2 + V_{y_0}^2 = V_0^2, \quad V_{y_0} / V_{x_0} = \tan\theta,$$

$$V_x = V_{x_0} \text{ remaining constant throughout}$$

while V_y de– or accelerates due to gravity (Galileo's *virtù di gravitá*).

Then, according to the law of conservation of energy, we must have the total

Kinetic + Potential at release (0,0) = Kinetic + Potential at peak (x_p,y_p)

$$\text{i.e. } \tfrac{1}{2}m(V_{x_o}^2 + V_{y_o}^2) + 0 = \tfrac{1}{2}m(V_{x_o}^2 + 0) + mgy_p$$

whence $y_p = V_{y_o}^2 / 2g$ or $V_0^2 \sin^2\theta / 2g$ (the masses cancelling out);

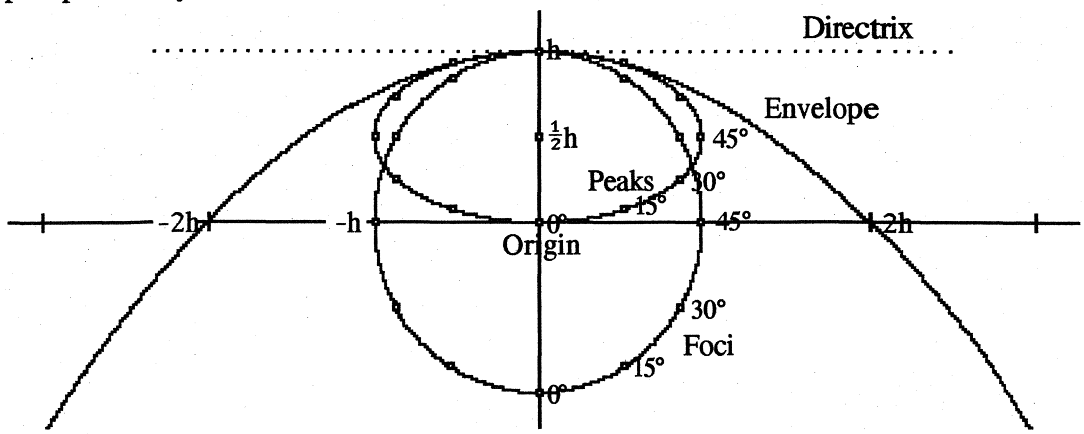
and since at peak also $V_{y_p} = V_{y_o} - gt_p = 0$

while any $x_n = V_{x_o} t_n$ where t_n is time for drop to reach (x_n,y_n) from (0,0),

it follows that $t_p = V_{y_o} / g$ and $x_p = V_{x_o} V_{y_o} / g$ or $V_0^2 \cos\theta \sin\theta / g$,

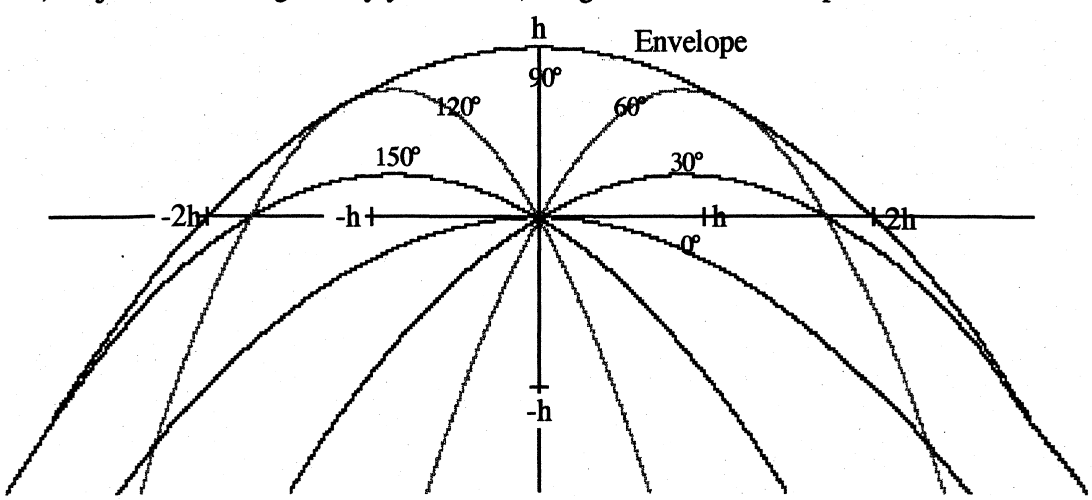
which for $\theta = 90°$ gives highest point $(x_p,y_p) = (0, V_0^2/2g) = (0,h)$.

This allows us to work graphically with the simple height h, and it is a straight-forward exercise (details of which are given on p. 5 of [5]) to show that the locus of all parabola peaks is an ellipse 2h wide ($\theta = \pm 45°$) and h high ($\theta = 90°$) centered at $(0, \frac{1}{2}h)$, with tangent $y = h$ as common directrix, focal points describing a circle $x^2 + y^2 = h^2$ centered at (0,0), and lastly the sought-for envelope a parabola $y = h - x^2/4h$, as shown in First View below.



First View

To describe the individual jets, we begin with the parametric equations $x = V_{x_o}t$ (constant kinetic) and $y = V_{y_o}t - \frac{1}{2}gt^2$ (kinetic accelerated downward), then substitute $t = x/V_{x_o}$ to obtain $y = V_{y_o}(x/V_{x_o}) - \frac{1}{2}g(x/V_{x_o})^2 = x\tan\theta - x^2/4h\cos^2\theta$ which can be shown to be a parabola open downward $y = h\sin^2\theta - (x - 2h\sin\theta\cos\theta)^2/4h\cos^2\theta = h\sin^2\theta - (x - h\sin 2\theta)^2/4h\cos^2\theta$ having peak at (x_q, y_q) and focus at $(x_q, y_q - h\cos^2\theta)$ lying respectively on the ellipse and circle claimed above. In particular, the jet at $\theta = 0°$ is given by $y = -x^2/4h$, congruent to the envelope but lowered h units.

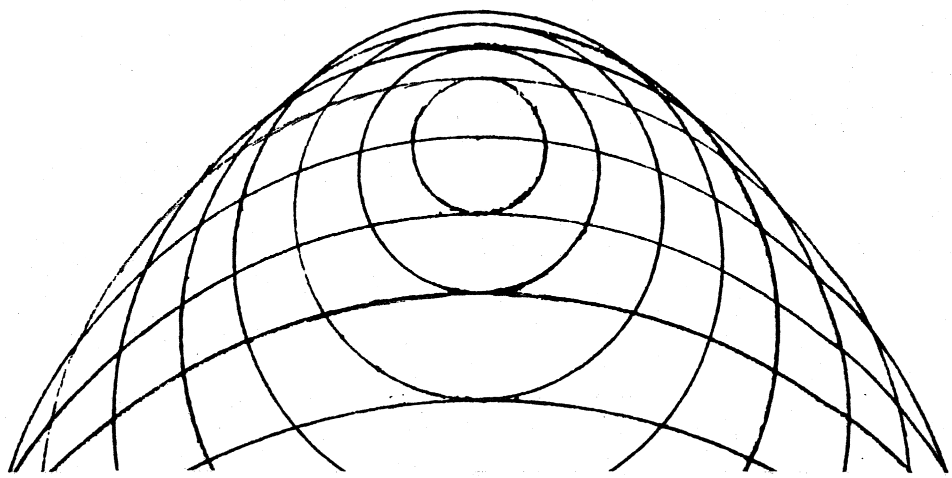


Second View

When the same parametric equations are expressed in terms of trigonometric functions as $x = V_{x_o}t = (V_o \cos\theta)t$ and $y = V_{y_o}t - \frac{1}{2}gt^2 = (V_o \sin\theta)t - \frac{1}{2}gt^2$ but the angles of release now eliminated by substituting $\sin^2\theta = 1 - x^2/V_0^2t^2$, we obtain the family of circles on next page given by equation

$$x^2 + (y + \tfrac{1}{2}gt_n^2)^2 = V_o^2 t_n^2.$$

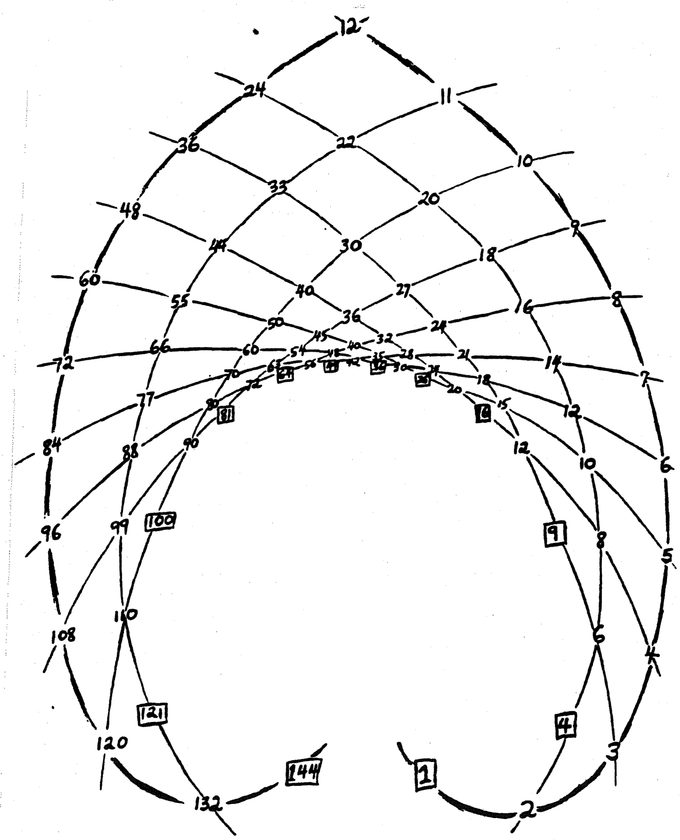
These expand linearly in radius while falling quadratically (acceleratedly) in time like fireworks.



Third View

2. The Quest for Alternative Natural Shapes for the Times Table

By coincidence, it was Dana Williams' father, Noah Williams III [6], on whose 4th grade blackboard at Highland Hall I discovered the following intriguing version of a times table which he had learned from one of his teachers, William Harrer, at the New York Rudolf Steiner School [10].

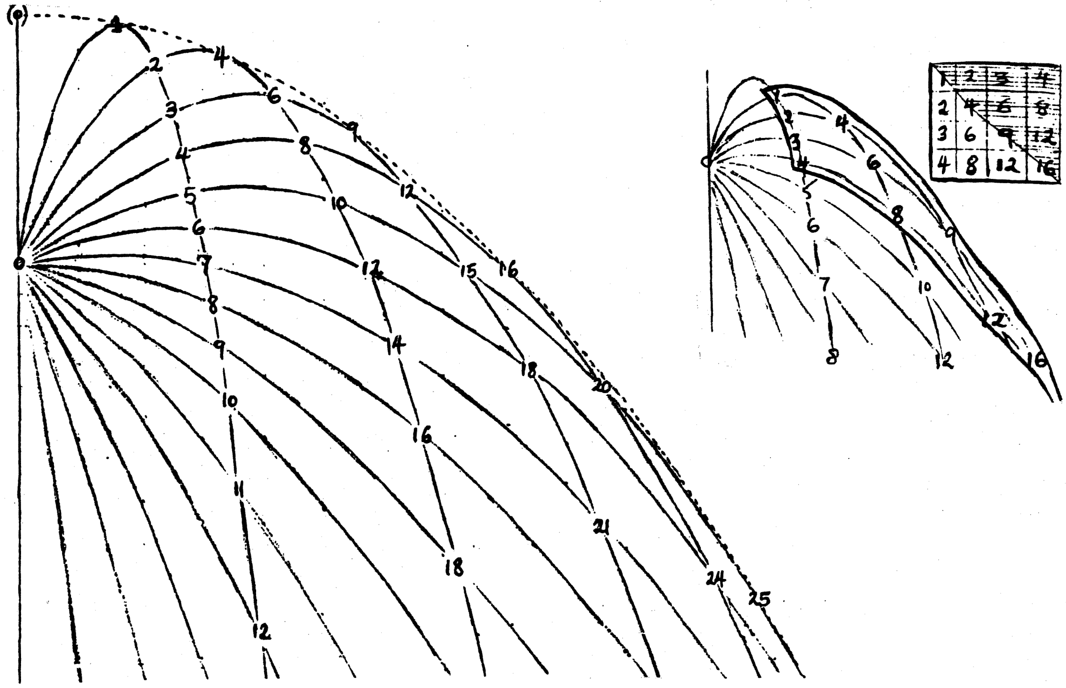


Called "The Great Times Table," it showed all the products from $1\times1 = 1$ up to $12\times12 = 144$, individual "rows" or "columns" of the usual square formatted table (such as 1 2 3 ···, 2 4 6 ···, and so on) following peculiar curves, with most products appearing at places where curves intersected (for example 12 at mid-level on right is where the vertical 2 times "column" curve 2 4 6 8 ··· meets the horizontal 6 times "row" curve 6 12 18 ···, so 12 is their product 2×6). Perfect squares 1, 4, 9, ··· up to 144, on the other hand, appear in mid-segments marked by square frames, and are all grouped around the central cavity.

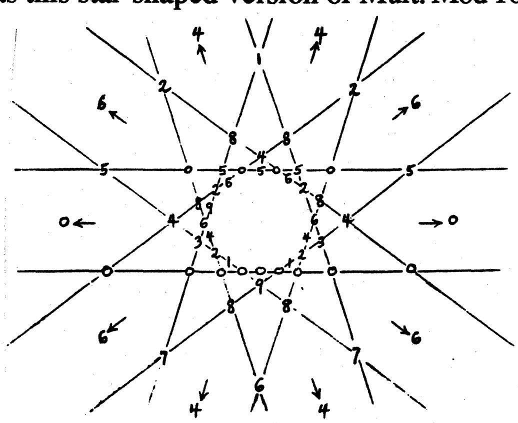
After some reflection, I realized that it was simply a 45°-45°-90° semi-square which had been bent around attractively if arbitrarily into a leaf or heart shape, the other half square having been deleted as redundant due to commutivity.

Imagine my surprise when these two apparently unrelated questions from two generations of the same family — the shape of a water fountain from Williams *fils* and the shape of a times table from Williams *père* — turned out to be related! A letter from another High Mowing alumnus, Christopher Stoney ([7], p. 7), showed how the right-hand half of the fountain could be labelled as a times table, its rows and columns now parabolic jets, general products appearing at their (profile) intersections, and perfect squares at their points of tangency to the over-all parabolic envelope.

The insert ([7], p. 8) shows how a section of the fountain corresponds to the usual semi-square. Notice that 0 appears in two places: Once at origin as common multiple of all other numbers (common source of all jets), and again at top-most point of envelope representing perfect square 0^2.

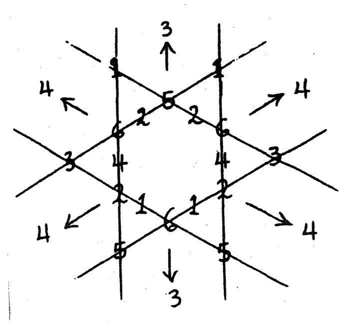


It is conventional to place 1 as identity element at the top corner of a multiplication table and omit 0, since while $0 \cdot n = 0$ is true for all n its very truth makes solving $0 \cdot x = 0$ for x impossible. The presence (let alone double presence) of 0 in the water fountain table therefore disturbed Mr. Stoney, and he began coming up with finite modular multiplication tables (in "clock arithmetic") with either lots of 0's such as this star-shaped version of Mult. Mod 10 which appeared in [7], p. 8:

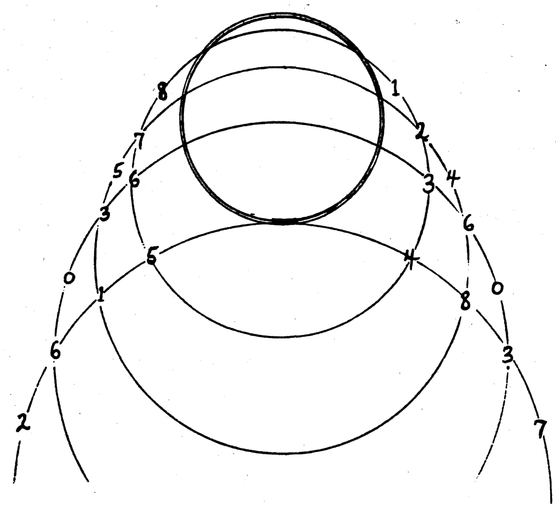


or this star-shaped version of table for usual zeroless Mult. Mod 7 which appeared in [8], p. 18:

1	2	3	4	5	6
2	4	6	1	3	5
3	6	2	5	1	4
4	1	5	2	6	3
5	3	1	6	4	2
6	5	4	3	2	1

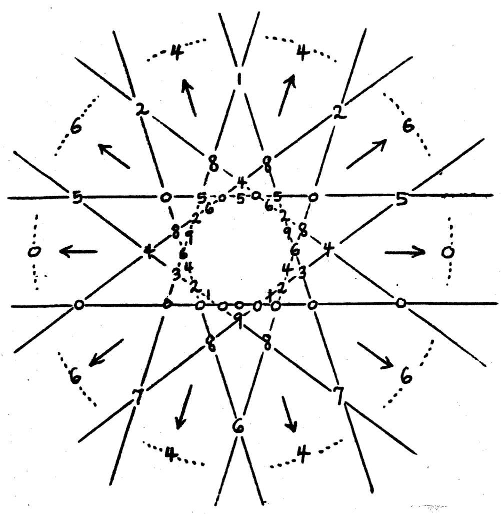


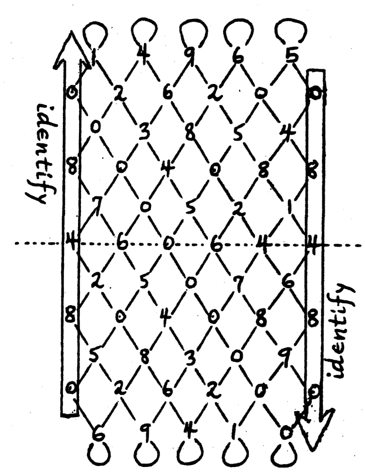
or, again with 0's, this version of Mult. Mod 9 ([7], p. 9) which cleverly uses the falling-circles version of the water fountain display to model its cyclically repeating "rows" and "columns"
0 1 2 3 4 5 6 7 8,
0 2 4 6 8 1 3 5 7,
0 3 6 0 3 6 0 3 6,
etc., with all of the 0's lying (as 0th "row" or "column") on the top-most circle (printed double-thick for emphasis) to be read as a giant "0."



[A word on the measurements used in falling-circles model on 4th p.: Taking $g = 32$ ft./sec.^{2}, it is convenient to take $h = 16$ ft. so that $V_0 = \sqrt{(2gh)} = 32$ ft./sec. The circles lie a $\frac{1}{2}$ sec. apart.]

Just when it seemed about everything possible had been done and this water/table discussion had run its course, a letter [9] arrived from Timothy Poston (then at Battelle Institute, Geneva) noting that each of these stars such as "the multiplication table mod 10 belongs on the Möbius strip.





This is visible from Stoney's diagram [repeated above left], which is implicitly drawn on the projective plane. I find it more natural to remove the hole in the middle (with no numbers on it) than the circle at infinity (with five): the [dotted] line on my table shows where Stoney's points at infinity go. The notation has exactly one line per number and a product [only] at each intersection."

Bibliography

[1] Claude V. Palisca, "Vincenzo Galilei," in Vol. 9 of *The New Grove*, 2nd ed., Macmillan, 2001.
[2] Hermann von Baravalle, *Das Reich geometrischer Formen*; Verlag Freie Waldorfschule, Stuttgart, 1935.
[3] Hermann von Baravalle, *Geometrie als Sprache der Formen*; Novalis Verlag, Freiburg im Breisgau, 1957; subsequent editions pub. by Verlag Freies Geistesleben, Stuttgart, 1963 and 1980.
[4] Col. Robert Stanley Beard, *Patterns in Space*, Creative Publications, Palo Alto, 1973, et al.
[5] Stephen Eberhart, "The Threefold Waterfountain," *Mathematical-Physical Correspondence* 13, Michaelmas 1975.
[6] Noah Williams III, "The Great Multiplication Table," *M.-P. C.* 11, Easter 1975
[7] Christopher Stoney, "Excerpts from Two Letters," *M.-P. C.* 13, Michaelmas 1975.
[8] Christopher Stoney, "A Zeroless Multiplication Table, *M.-P. C.* 14, Christmas 1975.
[9] Timothy Poston, [Excerpt from a Letter], *M.-P. C.* 14, Christmas 1975.
[10] William Harrer, "The Great Multiplication Table," *Education as an Art* Vol. 3 No. 2, 1942.