

Introducing Simple Arithmetic and Geometric Series in Complex Parametric Modeling

Jane Burry
Mark Burry
Spatial Information Architecture Laboratory
RMIT University
Swanston Street
Melbourne 3001 Victoria Australia
E-mail: jane.burry@rmit.edu.au

Abstract

Deferment is an important concept in design. Escaping from a strict hierarchy of inflexible decision-making can be important both in the quality of the result and in avoiding decision reversals and rework during design and construction. Engaged in a speculative activity, the designer never has access to all the information influencing any particular design decision. The object of design, whether building, artifact, art piece, film or computer game, as the confluence of many decisions and influences is intrinsically both complex and incomplete during the process. This paper will examine a detailed example of a way of structuring a digital design model geometrically to maximize its flexibility and the opportunity to tune geometrical decisions late, in the light of as many different factors as possible. Paradoxically, this example employs a rigidly hierarchical system of organization employing a tree structure that avoids lateral links and “evolutionary dead ends” in order to work effectively. Thus one rigid hierarchy of decision making at meta-design level replaces another to achieve deferment and flexibility at formal design level. In architecture, the opportunity to defer or suspend precise dimensions and relationships by substituting for these with parametric relationships has proven advantages for collaboration. These manifest as unparalleled precision in construction, time- and cost- savings at the production end of the process. At a more conceptual level, parametric modeling also provides the opportunity to experiment geometrically at comparatively little cost in terms of rework. The current work in progress on the upper colonnade of the west transept of Antoni Gaudí’s Sagrada Família church begins to introduce a new level of interest into the parametric tree, an opportunity through equation driven parametric relationships to generate form not only through defined geometric relationships but to introduce variable finite and infinite series into these relationships.

The Significance of Arithmetic and Geometric Sequence and Series in Architecture and Natural Growth and Form

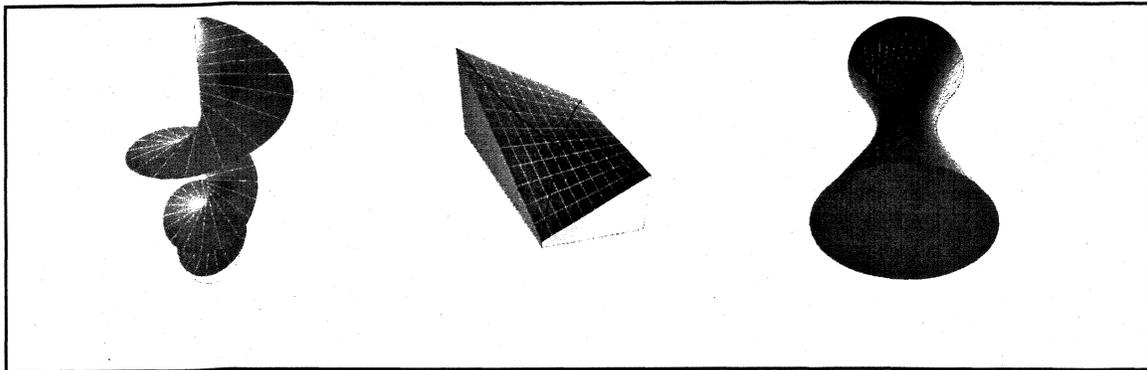
The Fibonacci sequence is probably the best-known mathematical sequence found widely in both natural process and form and in architectural form making. This is the sequence 1,1,2,3,5,8,13, ..., where the n th term is the sum of the $(n-1)$ th and the $(n-2)$ th and the ratio between consecutive terms tends to the golden number Phi (approximately 1.618) or its reciprocal phi (approx 0.618). This proportion is manifest in formal patterns of growth, the spirals seen in seed heads, where the irrationality of Phi provides the best opportunity for the distribution and close packing of seeds, associated leaves and petals, as they grow. The golden proportion, known to the Greeks, identified within the Platonic solids, defines the proportions of the rectangles linking the centre points of the faces of a regular dodecahedron or vertices of an icosahedron. In architecture it is widely found in proportional systems and the proportions of single elements of classical architecture. This is also applied in the proportioning grid developed as the basis for architecture based on human scale by Le Corbusier: the Modulor. [1]

Another example of the significance of sequence and series in the construction and comprehension of space is linear perspective. Filippo Brunelleschi’s famous optical experiments with the baptistery in Florence provided the rules governing perspective lines converging at the horizon used to structure the two dimensional representation of three dimensional space and objects, and the modern understanding of the phenomenon. Perspective is a useful tool with which to manipulate perception, a subtle preoccupation highlighted by the Greek use of entasis, the slight curvature applied to the linear profile of a column in order to make it appear straight. We detail below part of a model constructed to be able to manipulate these effects with subtlety through the spacing, distribution, scale

and inclination of serial elements in the crowning colonnade of the Passion façade of the Sagrada Família church.

Sagrada Família Church Background

Antoni Gaudí worked on the design of the Sagrada Família church for forty-three years or almost the full span of his working life. At the time of his death in 1926, the only parts of the church that had been built were the crypt, the apse and most of the towers of the east transept or Nativity façade. For the last 12 years of his life he worked exclusively on the church, ultimately living cloistered on site. It was clear at this time that he would not oversee the whole of the construction of the church during his own lifetime and that the intentions for its complex form would have to be resolved, expressed and communicated unequivocally to subsequent generations of collaborators. This contrasts strongly with his role in the construction of the crypt for the Colonia Güell church, designed from the structural hanging models, the detailed execution was nevertheless the product of Gaudí's own active collaboration on site in discussion with the craftspeople at work with some scope for the reactive and spontaneous intervention. The lineage within the Sagrada Família itself indicates that the design developed over the period of involvement. There is an obvious metamorphosis from the Gothic revival idiom of the crypt, apse (both commenced before Gaudí took over the commission from his predecessor del Villar in 1883) and lower lateral nave designed by Gaudí to the much more fluid use of form in the stone of the upper lateral nave and central nave windows [2]. During the period of his involvement, Gaudí developed and refined a codex, a family of forms based on the geometry of second order ruled surfaces and an approach to equilibrated structure that eliminated the need for the buttresses, flying buttress and pinnacles, "parasitic superstructures of dead weights and counter-thrusts" used to verticalise the thrust in the Gothic cathedral [3]. The three ruled surfaces used throughout the church are the helicoid, the hyperbolic paraboloid and the hyperboloid of revolution of one sheet. [Figs. 1-3] All three are highly generic; the hyperboloid, for example, is defined by a total of nine variables, three defining its translational location, three defining its rotation about each of three Cartesian axes, two defining the major and minor axes of the elliptical throat and one defining the steepness of the asymptote. Compared to a spherical surface, for example, which can be defined simply by the three coordinates of its position in space and a single radius, the hyperboloid presents a diverse family of potential surfaces.



Figures 1-3: *The helicoid, hyperbolic paraboloid and hyperboloid of revolution of one sheet showing their generation from straight lines*

The three types of surfaces can be generated graphically, using straight lines. The helicoid is created by rotating a straight line around the central axis while its centre of rotation is translated along the axis. The ratio of the angle of rotation to the corresponding distance of translation of the centre will determine the shape and curvature of the resulting surface. A portion of a hyperbolic paraboloid can be generated conceptually by taking a quadrilateral (rectangle or parallelogram), generating grid lines parallel to the two directions of the sides of the figure, and then moving one of the four corner vertices out of the original plane of the figure. All the grid lines and the edges remain straight lines but the surface between them is now a doubly curved saddle form. Finally, a portion of a hyperboloid

can be generated by describing two circles, or more generically, two ellipses on different planes, drawing a line from a point on one ellipse to a point on the other ellipse and moving the ends of this straight line uniformly about both figures so that one complete revolution occurs simultaneously on both ellipses. If the two circular or elliptical figures are similar, and similarly oriented on parallel planes, and the line is parallel to the axis joining the two centers, the surface generated by the sweep of the line will be a cylinder or elliptical extrusion. If, for the same conditions, the line joins two opposite points and passes through the axis joining the two centers, the surface will be a circular or elliptical cone. For every other line linking the two ellipse edges, the surface generated by its revolution is a hyperboloid, the form commonly associated with power station cooling towers.

Thus, each of these surfaces can and has been modeled in gypsum plaster using end templates and straight edges linking points on the templates and dragged through the wet plaster. They can be represented in two-dimensional documentation, again by describing the shape of the edge templates and specifying their separation, inclination and the particular points on each template linked by straight lines. Finally, they can be built at full scale, once more taking advantage of their straight line description. The stone mason with two end templates with points and corresponding points marked can chisel straight lines between them to generate a good approximation of the surface for finishing. Robot stonecutters now extend this technique. The nave walls and vaults of the Sagrada Família church have been constructed from artificial stone, in this case using the straight lines on the surface to construct the moulds, with reinforcing bars also meshing along the generatrices of the surface. The hyperboloid is defined by two sets of opposing straight lines, seen in the lines of reinforcement for the vaults.[Fig. 5]

The apparent simplicity of the geometrical generation of single second order surfaces is belied by the complexity of combining them and finding and manipulating the fourth-order curves of intersection between adjacent surfaces. This has been a particular focus in unraveling the surviving evidence of Gaudí's design intentions and working from his 1:25 and 1:10 scale plaster models up to fully resolved 1:1 precision information required today for construction. The church site was raided during the Spanish Civil War; drawings burnt and plaster models smashed. For many years subsequently the principal work for Gaudí's immediate successors was to collect, identify and catalogue model fragments so that the models could be painstakingly reconstructed and reproduced. From measurements taken from the plaster models, manual graphical techniques were used to reproduce the profiles and generatrices of the curved surfaces. Superimposed drawings in each of the three Cartesian planes would then be used to identify and test the intersection between adjacent surfaces. As can be seen in Figure 4, the intersection and particularly the triple points of intersection between three surfaces are the critical elements of the composition. Having established the intersections and produced the plan section and elevational representations of these, to then make changes, the whole graphical process would need to be repeated using informed hypotheses for the altered values for the new hyperboloids and checking the new locations of the intersections. At the time when transferring this lengthy manual graphic process to digital computation was first investigated, it was clear that to do more than replicate it, a very different approach to design was required as well as the power of high-end computers and software.



Figure 4-5: *The plaster model of the nave window showing the straight line on the surface and the nave vaults under construction showing the application of the straight line geometry in the Catalan tile vaulting and in the placing of reinforcement*

Parametric Design in Architecture

In working from points measured on reconstructions of the surviving Gaudí models, the problem becomes one of working backwards to establish and locate mathematical surfaces in the computer model that not only closely match the measured points but achieve curves and triple points of intersection that accord with the overall compositional intentions for the assembly. Clearly, with nine variables per surface and numerous surfaces in a single window composition, this becomes a problem that can only be solved iteratively. A change to one variable for one surface (a small rotation about one of its local axes, for instance), can result in a radical change in the curve of intersection with an adjacent surface. Changes that appear to make little impact in the composition of the elevation can nevertheless impact significantly on the position of intersections in the depth of the wall or perpendicular to the notional neutral plane. Parametric design has provided a way of structuring the model where rather than trying to find an optimal geometric solution for each individual surface, and then combining them with unpredictable results, it is possible to model the whole composition (or half the composition, as each of the window assemblies is symmetrical about a central plane). Relationships are created between the surfaces but these and the precise shape and inclination of the individual surfaces is kept fluid by assigning variable parameters to all the numbers determining these [4].

Work in Progress: Serial Interpretation

The design of the lateral nave, central nave windows, the nave columns, the central nave window emblems, the columnettes of the triforium, the desaignues, the nave roof and the west transept rose window of the Sagrada Família have all presented particular issues and required particular parametric approaches to unlock the geometries represented in the models. In particular the need to model what is not there, whether by modeling forms that are removed from a notional wall by Boolean subtraction or by understanding the geometry as part of a greater whole where the un-built virtual component contributes the critical information to model the built fabric [5].

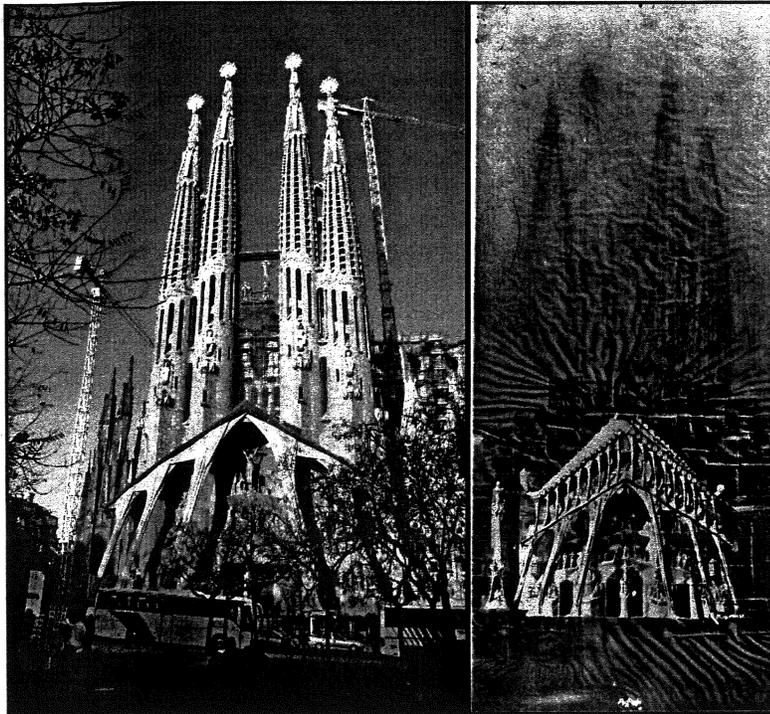


Figure 6-7: Contemporary photograph of the west transept of the Sagrada Família church showing the site for the colonnade and gable next to 1917 photograph of the drawn proposal for the elevation

Current work in progress on the upper colonnade and gable of the west transept also presents a new challenge. While previous work has called for the definition of a bay of the nave that is then repeated, there is a non-repetitive serial pattern in the application of the repeated elements of the colonnade and gable that invites interpretation. The openness of interpretation is accentuated by the absence of surviving original plaster models for this part of the church. The principal original source is a photograph of a drawing dated 1917 of the west elevation. All other material: drawings and a plaster model constructed initially under the direction of Puig I Boada in the 1970s and subsequently by Cardoner in the 1980s, is posthumous and interpretative. There is additional information from other parts of the church bordering on this area and there is also now a measurable and detailed "site" defined by the porch and cornice already built in support, the west towers behind and the now-completed rose window between the towers that will be framed by the colonnade. The image of the colonnade in the photograph portrays a series of subtly and organically varying bone-like columns framing a stair that leads to a high platform above a concealed selpulchra: the cave or tomb. The back wall of the colonnade contains niches to house figures of the saints. The whole of the west façade is dedicated to the Passion. While the porch below graphically portrays the crucifixion through the confronting sculpture of Subirachs, the upper portion makes allegorical reference to the tomb. There is a sense in which the individual columns relate in the same way as the individual bones in the ribcage, increasing subtly in girth and length to suit their position in the whole array. The trimmed hexagonal prisms above them, spanning between the colonnade and the back wall, clearly relate in their distribution to the intercolumniation but increase in height in a linear relationship to the horizontal distribution across the front plane. The steps of the cornice above also increase in both height and length at a greater rate than the hexagonal prisms below them but the relationship of the steps to the hexagonal prisms is less clear than the distribution of the prisms to columns. The whole begins to read like a musical score with numerous voices and instruments probably within one time signature but with very varied parts and much-concealed notation within the individual lines. Add to this two dimensional reading in elevation, the third dimension, depth, curvature in plan and the potential for visual rhythm become very rich. How can this wealth of possibility be structured geometrically, and ultimately numerically in a way that allows the manipulation of its numerous components without over constraint or premature commitment to any particular reading? By describing in detail part of the parametric schema currently underlying this model, we will illustrate the opportunities generated for deferment and serial ordering. At the same time it will illustrate the current limitations encountered with the approach.

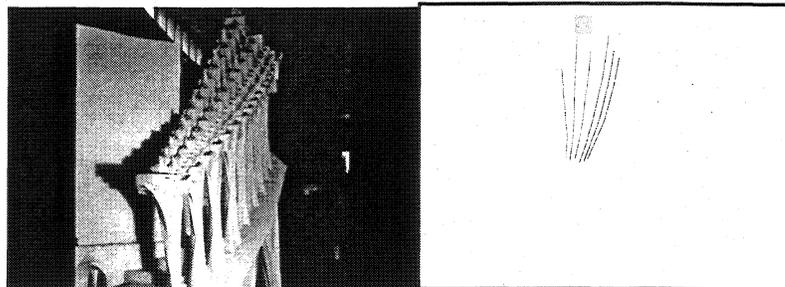


Figure 8-9: *Cardoner's plaster model of the colonnade and gable completed during the 1980s and curves extracted from a digital facsimile of the model for analysis*

Parametric Schema for the Colonnade: Detailed Example

As a starting point for the model, an explicit curve from the plaster model and the equivalent two-dimensional curve from the scanned photograph were imported into the model space. This was the curve linking the top front apexes of the trimmed hexagonal prisms. These curves were not used directly in the model except as a guide to an initial location for the first two parametrically defined points now created, one at the lower end of the curve and one at the upper end, close to the center of the gable. The parametric points were projected onto the xy plane (parallel to the ground) and onto the yz plane (front) separating the curve into a straight line in elevation and a curve in plan. The intention was to be able to control the curvature, the inclination of the line in elevation and the location of the

end points. In the front plane, the point was used to locate two new parametric planes, one parallel to the xy (base) plane and the other at a variable angle to this one about an axis through the lower projected point and perpendicular to the front plane. A vertical line was constructed through the higher projected point, and a new point created at the intersection of the inclined plane with this line. Finally, a line was defined by two endpoints, one the lower projected point (derived from the projection of the initial point with variable coordinates) and the second, the point of intersection between the inclined plane and the vertical line. The whole of this first section of the tree set up a control mechanism for the position and the inclination of the line governing the array of hexagonal prisms as viewed from the front (yz view). So far this small version of “The House That Jack Built” only constrains the hexagons to lie on a straight line as viewed in front elevation. The line can start and finish anywhere (controlled by the variable coordinates of the first two parametric points) and can be as steep as we wish (controlled by the angle between the two planes constructed on the projected lower point). In other words, there are seven numbers controlling this component, three coordinates for each of the two points plus the angle of inclination.[Figures 10-11]

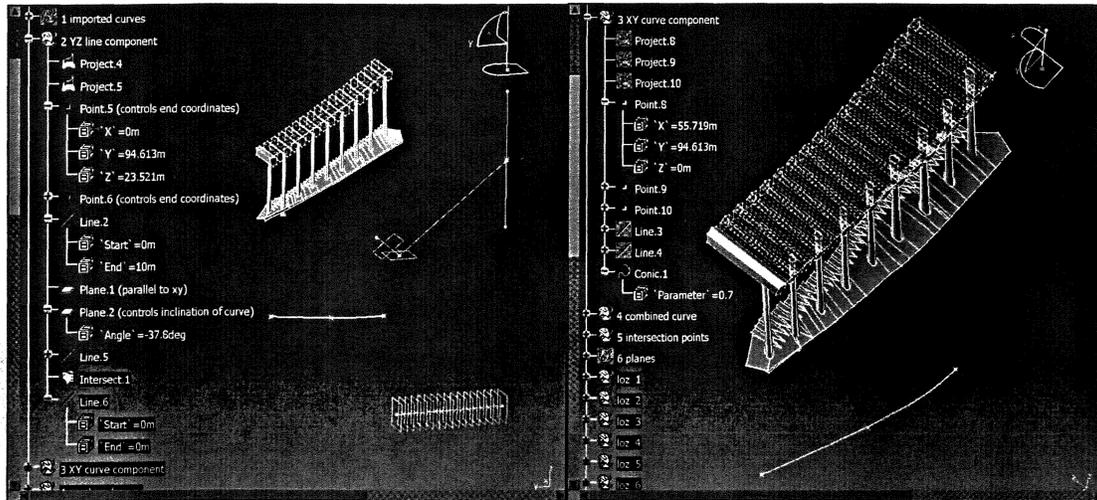


Figure 10a-10b: Images from the CATIA model showing the construction of the three dimensional curve from a variable straight line in elevation and a variable parabola in plan and the subdivision of the curve by planes normal to its projection onto both base and front plane.

The points projected onto the xy plane (parallel to ground) were used as the endpoints for a parametrically constructed parabola, comparing it to the imported explicit curve from the plaster model to create its initial shape. The decision to define this curve as a parabola was taken in reaction to curve analysis of curves extracted from the topographical information from the site and from the Cardoner plaster model. As noted above, it was not possible to find definitive information from an original source for the curvature of the composition in the third dimension; in this respect, the decision was taken to follow the interpretation of Gaudí’s earlier successors, some of whom had been involved during his lifetime, and work from an analysis of the growth sequence of the stepping gable which appears to be quadratic. The decision to define the curve as a parabola has great significance for the parametric model. While this curve can now be a portion of more or less any parabola through varying the coordinates of the parents of its endpoints and altering the parameter for its curvature, it cannot become another curve such as an arc or an ellipse except by deleting the subsequent model right back to this point as all that follows will have geometric dependency leading back to this point in the tree. It is impossible to substitute one type of entity for another; the definition of each element and the nature of its variability and dependencies need to be considered at the time it is created.

The part parabola in the xy (base) plane was now projected onto and normal to the yz (front) plane as a straight line. This straight line was then divided into sixteen equal parts representing the sixteen intervals between the seventeen adjacent hexagonal prisms. The decision to construct the model with seventeen trimmed hexagonal prisms (the prisms are trimmed by an inclined hyperbolic

paraboloid: the curved front face) accorded with the 1917 photograph [6]. It also allowed a regular relationship between the “hexagons” and the columns supporting them with a column shaft meeting the center of every second hexagon. The plaster model departs from this relationship. There are fewer hexagons and a relationship between the columns and hexagons is much less obvious. With regards to the 16 divisions of the line, each of the two endpoints and the fifteen intermediate points would locate the “center line” of a “hexagon”. (“Hexagon” is here in parenthesis because the six front edges of the trimmed prisms are in fact non-coplanar, comprising four straight line segments (directrices for the hyperbolic paraboloid (hp)) and two curves, the result of trimming with the hp surface.)

The division of the line creates fifteen new points each of whose position is equation-driven as a proportion of the total length of the line. Should we decide to return to the fifteen prisms of the plaster model as opposed to the seventeen in the photograph, it would be possible to delete two of the points and all their children (geometry built from this particular point) and adjust the equations to redistribute the remaining points. Similarly, it is possible to add points subsequently and define their position by a similar equation. However, in practical terms, this becomes another point of no return, particularly as we subsequently choose to create dependency between the column geometry and the prism geometry.

Here is an opportunity to introduce sequencing into the model. Suppose that instead of sixteen equal divisions, the width of the prisms and the corresponding spacing between them should increase in linear sequence by ten percent from the lowest to the highest. As this still needs to be expressed as a proportion of the total length of the line for each of the divisions, the equations become, for example:

$$\text{Length parameter for nth division point} = \text{total length of line} * (n(np + 2N)) / (N(Np + 2N))$$

This equation is derived from considering the series of arithmetically increasing prism widths and dividing the distance along the line to the nth point (centre of the nth hexagon) by the total distance to the Nth point (the centre of the final hexagon.)

L denotes the total length parameter of the line

N denotes the total number of intervals, in this case 16.

p denotes the total proportional change between the width of prism 0 with its centre at an endpoint and the width of prism N with its centre at the other endpoint, in this case 0.1.

The equations in the parametric tree appear in the form $(L / (Np + 2N)) * (2n + n^2p/N)$:

$$\text{length parameter for division point} = (\text{total length of line} / 33.6) * (2 + (0.1/16)) \text{ (1}^{\text{st}} \text{ point)}$$

$$\text{length parameter for division point} = (\text{total length of line} / 33.6) * (4 + (0.4/16)) \text{ (2}^{\text{nd}} \text{ point)}$$

$$\text{length parameter for division point} = \text{total length of line} / 33.6 * (30 + (22.5/16)) \text{ (15}^{\text{th}} \text{ point)}$$

This is a very simple and formally subtle example. The same approach to introduce a geometric sequence or a Fibonacci sequence could produce more dramatic results. Having set up this first set of fifteen equations, these are then used in this model to generate other equations similarly related to the sequence.

Seventeen planes are created normal to the divided line each passing through an endpoint or a point of division. The parabola in the xy plane and the straight line in the yz plane are combined to define a three dimensional curve. The seventeen planes are intersected with this resultant curve and these points of intersection are the points at which copies of a parametric model of the prism are introduced. The equations are now created that link the parameters established within the individual prism models back to the equations and parameters determining the distribution of the prisms. Thus, it is possible to control the prism dimensions as a function of their distribution, in this case linear growth.

The modeling of the individual prisms and columns will not be described in detail here, nor will the hierarchy of relationships controlling the columns. It is sufficient to state that the columns are a hybrid of hyperboloid shaft and hyperbolic paraboloid lateral elements. The programming required to combine these geometries has yet to be integrated into parametric software so the columns are represented in the parametric model by their hyperboloid shafts only. The throats of the hyperboloids are also distributed on a parametrically driven parabolic curve in plan and a straight line in front elevation. Their central axis intersects the lower edge of the associated prism, and this point of intersection can slide along the edge, affecting the inclination of the column in both yz (front plane) and xz (side plane). The serial distribution of the columns is linked to the equation-driven distribution

of the prisms and like the prisms, the size and proportions of each column can potentially be linked to their serial distribution, or kept independent, or independently serialised through a set of equations controlling the parameters for lengths of the major and minor axes of the cross-sectional ellipses generating the hyperboloid columns.

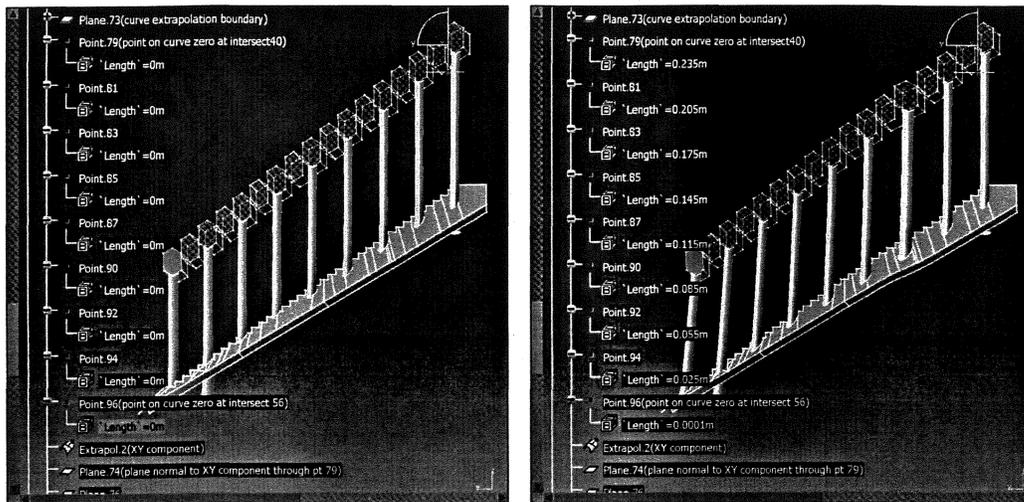


Figure 11a-11b: Image from the CATIA model showing an effect of altering the parameters governing the positions of the column throat centre points on the parabolic curve linking them. In this configuration, they are vertical view when the value is "0".

Conclusion

Although the original evidence for the design and construction of the upper colonnade and gable of the Passion façade is scant, its highly organic nature, the subtle sequencing and variation in its composition are clear even from the surviving photograph of the elevation. To achieve this, a highly flexible model has been constructed that allows almost unlimited experimentation in manipulating the proportions, curvature, inclination and distribution of the parts and their interrelationships. This is being used as a collaborative design tool between the academic research team that has built it and the technical team on site at the Temple Sagrada Família. While, on this same project, this flexibility in modeling has in the past provided the means of fitting precisely defined geometric surfaces to an agreed compositional strategy, its application is here extended, through an extended tool kit, to experimentation with serial relationships.

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