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NEC Polygonal Groups and Tessellations *

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Abstract

A kaleidoscope is obtained as the quotient of a space by the discontinuous action of a discrete group of transformations; this can also be obtained from a fundamental domain which characterizes it. In the present study, the specific case of the Hyperbolic Plane is analyzed with respect to the action of a hyperbolic polygonal group, which is a particular case of an NEC group. Under the action of these groups, the hyperbolic plane is tessellated using tiles with a polygonal shape. The generators of the group are reflections in the sides of the polygon. Clear examples of quadrilateral tessellations of the hyperbolic plane with Saccheri and Lambert quadrilaterals -designed using the *Hyperbol* package created for *Mathematica* software- and are found in the basic structure of some of the mosaics of M.C. Escher.

1. Introduction

A group that acts on its own and discontinuously in space, in general, defines a domain that characterizes it. Similarly, it would characterize the quotient of the space under the action on the group, in which case it is called a kaleidoscope. The center of our interest in the specific case of the hyperbolic plane under the action of NEC (*Non Euclidean Crystallographic*) groups, which are subgroups of the group of non-Euclidean isometries that act on their own and discontinuously upon the hyperbolic plane ([12]). We shall study polygonal groups, a particular kind of NEC groups, in order to tessellate the hyperbolic plane using tile with a polygonal shape and reflections on the sides of these polygons as generators of the group. The connection between this mathematical formulation and the artistic construction of geometric designs is based on the M.C. Escher's graphic works, who developed unique techniques for creating mosaics in the hyperbolic plane. We will show some examples of tessellations with these polygonal groups, on the basis of which this artist made his *Cirkellimiet* series. We also include an example done on the Poincaré half-plane, though we do not develop the mathematical tool underlying it. Readers interested in the Poincaré models of the hyperbolic plane may consult [3].

2. NEC Groups and Fundamental Regions

Let X be the hyperbolic plane with the topology induced by the hyperbolic metric. The model we shall use is the open disc unit of Poincaré D^2. Let Γ be a subgroup of the group of hyperbolic isometries of X, $Iso(X)$. The action of Γ on X is the natural application $\Phi : \Gamma \times X \longrightarrow X$, given by $\Phi(\gamma, z) = \gamma(z)$. For each $z \in X$ we denote the set $\Theta = \{\gamma(z) : \gamma \in \Gamma\}$ as the orbit of z by the action of Γ upon X. The action of a subgroup Γ on X gives rise to the following relationship:

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$$\forall z, w \in X,\ z \backsim w \Leftrightarrow \exists \gamma \in \Gamma : w = \gamma(z).$$

This relationship $\backsim$ is an equivalence relation on X, whose equivalence classes are the orbits of the elements of X by the action of Γ. The quotient set $X/\backsim$ will be designated as $\Gamma \setminus X$ and is called the orbit space.

Definition 1 Let Γ be a subgroup of $Iso(X)$. Γ is said to be an NEC group, that is, a Non-Euclidean Crystallographic Group, if it is discrete (with compact open topology) and the orbit space is compact.

If Γ is an NEC group, then its rotations have multiple integer angles of $2\pi/n$, with $n \in \mathbb{N}$, and they do not contain limit rotations that is, rotations whose center lies on the infinity line.

The NEC groups are classified in agreement with their signature ([12]), which has the form:

$$(g; \pm; [m_1, ..., m_r], \{(n_{11}, ..., n_{1s_1}), ..., (n_{k1}, ..., n_{ks_k})\})$$

where the numbers m_i (periods) and n_{ij} (period cycles) are integers greater than or equal to 2, and g, r, k are non-negative integers. This signature determines a canonic presentation of the group Γ given by the generators:

$(i)\ x_i$	$i = 1, ..., r$	
$(ii)\ e_i$	$i = 1, ..., k$	
$(iii)\ c_{ij}$	$i = 1, ..., k$	$j = 0, ..., s_i$
$(iv)\ a_i, b_i$	$i = 1, ..., g$	(if the sign is $+$)
$(v)\ d_i$	$i = 1, ..., g$	(if the sign is $-$).

and the relationships:

$$x_i^{m_i} = 1$$
$$e_i^{-1} c_{i0} e_i c_{is_i} = 1$$
$$(c_{i,j-1})^2 = (c_{i,j})^2 = (c_{i,j-1} c_{i,j})^{n_{ij}} = 1$$
$$x_1 ... x_r e_1 ... e_k a_1 b_1 a_1^{-1} b_1^{-1} ... a_g b_g a_g^{-1} b_g^{-1} = 1 \qquad \text{(if the sign is } + \text{)}$$
$$x_1 ... x_r e_1 ... e_k d_1^2 ... d_g^2 = 1 \qquad \text{(if the sign is } - \text{)}.$$

The x_i isometries are elliptic, the e_i isometries are generally hyperbolic, the a_i, b_i are hyperbolic, the c_{ij} are reflections and the d_i are glide–reflections ([12]).

The fundamental regions comprise the smallest tile that allow us to tessellate by means of the action of an NEC group. For this purpose we establish the following definition.

Definition 2 ([3]) Let $F \subset X$ be closed and Γ an NEC group. F is said to be a *fundamental region* for Γ if:

i) for each $z \in X$ there exists $\gamma \in \Gamma$ so that $\gamma(z) \in F$

ii) if $z \in F$ is such that $\gamma(z) \in F$, with $\gamma \neq 1$, then $z, \gamma(z)$ are in boundary of F

iii) $F = Closure(Interior(F))$.

It is important to point out that this definition is an improvement upon those used in [10] and [12], in the sense that we eliminate the cases of *pathological* regions having isolated points that correspond with other points of the fundamental region under consideration.

3. Tessellations of the hyperbolic plane

Definition 3 Let Γ be a NEC group. A Γ-tessellation of the hyperbolic plane X, with the fundamental region F is a set of regions $\{F_i\}_{i\in I}$ such that:

i) $\bigcup_{i\in I} F_i = X$

ii) $int(F_i) \cap int(F_j) = \varnothing$ for $i \neq j$

iii) $\forall i \in I, \exists \gamma \in \Gamma : F_i = \gamma(F)$.

In the study of subgroups of NEC groups, as well as int he tessellations of X, the following proposition is important:

Proposition 1 ([11]) Let Γ by an NEC group and Γ' a subgroup of Γ with the index p ($|\Gamma : \Gamma'| = p$). Let F^{Γ} by a fundamental region for Γ. Then Γ' has a fundamental region $F^{\Gamma'}$ resulting from the union of p congruent replicas of F^{Γ} and besides $\mid \Gamma \mid \cdot p = \mid \Gamma' \mid$.

From F^{Γ} the fundamental region obtained for Γ' has the form

$$F^{\Gamma'} = \bigcup_{i=1}^{p} \gamma_i F^{\Gamma},$$

there the isometries $\gamma_1, \gamma_2, ..., \gamma_p$ have been chosen adequately in each of the classes of Γ/Γ'. That is , $F^{\Gamma'}$ can be construed from F^{Γ} by successively joining it at the boundary to $p-1$ $\gamma_i F^{\Gamma}$ copies. Furthermore, if F^{Γ} is connected, the $\gamma_1, \gamma_2, ..., \gamma_p$ can be chosen in such a way that $F^{\Gamma'}$ is also connected ([11]).

Polygonal groups are those that are generated by reflections in the sides of a polygon. Certain hyperbolic polygons give rise to NEC groups, and the hyperbolic plane is tessellated with them.

Lemma 1 ([1]) Let $\alpha_1, \alpha_2, ..., \alpha_k$ be real numbers with $0 \leq \alpha_i < \pi$ for each $i = 1, 2, ..., k$. Then, there exists a convex hyperbolic polygon F with angles $\alpha_1, \alpha_2, ..., \alpha_k$ if, and only if,

$$\sum_{i=1}^{k} \alpha_i < (k-2)\pi.$$

To effect tessellations it is practical to bear in mind that the F polygons described in the above Lemma have the following properties:

a) the bisector angles of F are concurrent in a point O

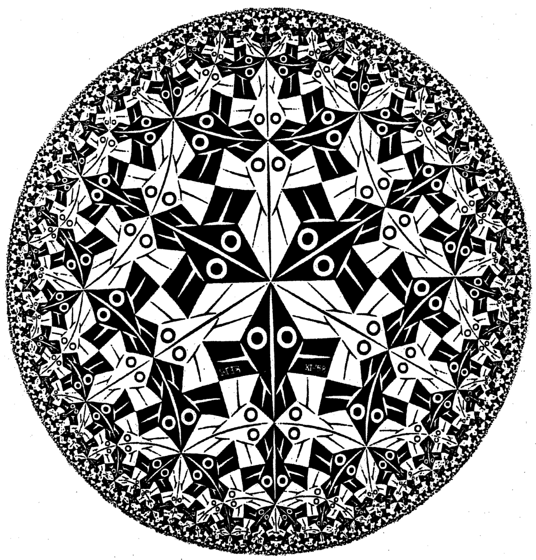
b) there exists an inscribed disc with its center in O that touches all the sides of F.

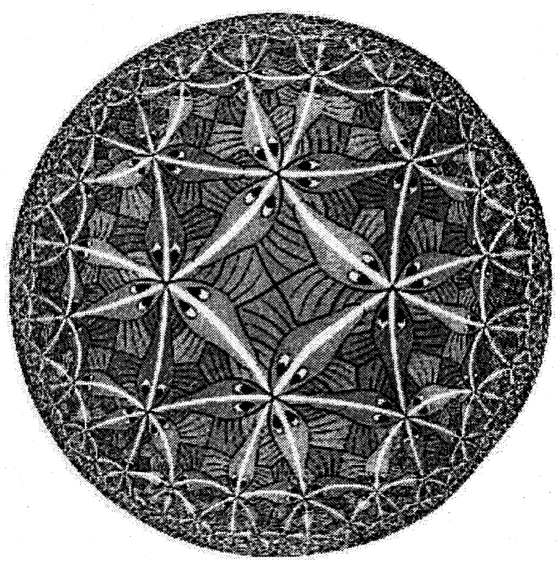
Theorem 1 ([10]) Let Γ be an NEC group. Then, there exists a convex polygon F that constitutes a fundamental region for Γ.

The following Theorem can be considered the inverse of the previous Theorem when the group is polygonal. The method of proof used is noteworthy – it is based on the covering of the hyperbolic plane by means of a family of sets that fit together, constructed from finite unions of set with the orbits of the points contained in the covering. A demonstration of this can be seen in [5].

Theorem 2 Let F be a convex polygon of k sides and interior angles π/n_i, with $n_i \in \mathbb{N}, n_i \geqq 2$, for each $i = 1, 2, ..., k$, with vertices P_i and sides determined by the segments $P_{i-1}P_i$ where $P_0 = P_k$, satisfying $\sum_{i=1}^{k} \pi/n_i < (k-2)\pi$. Let σ_i be the reflection on the line containing the segment $P_{i-1}P_i$ for each $i = 1, 2, ..., k$. Then the group Γ generated by the reflections $\sigma_1, \sigma_2, ..., \sigma_k$ (polygonal group) is an NEC group. Moreover, F is a fundamental region for Γ, with the signature $(0; +; [-]; \{(n_1, n_2, ..., n_k)\})$, and whose representation is given by the following relationships: $\sigma_i^2 = 1, \forall i = 1, 2, ..., k$, $(\sigma_{i-1}\sigma_i)^{n_{i-1}} = 1, \forall i = 1, 2, ..., k$, with $\sigma_0 = \sigma_k$ and $n_0 = n_k$.

In addition, the family $\{\gamma(F) : \gamma \in \Gamma\}$ constitutes a Γ-tessellation of X.





Example 1 Tessellation of the Poincaré disk constructed by using a polygonal group: M.C. Escher's "Circle Limit I" (*Cirkellimiet I*) ©2003 Cordon Art B.V.- Baarn - Holland. All rights reserved.

We can see polygons in the tessellation M.C. Escher's "Circle limit III" (*Cirkellimiet III*) ©2003 Cordon Art B.V.- Baarn - Holland. All rights reserved. It not is a constructed tessellation by using a polygonal group.

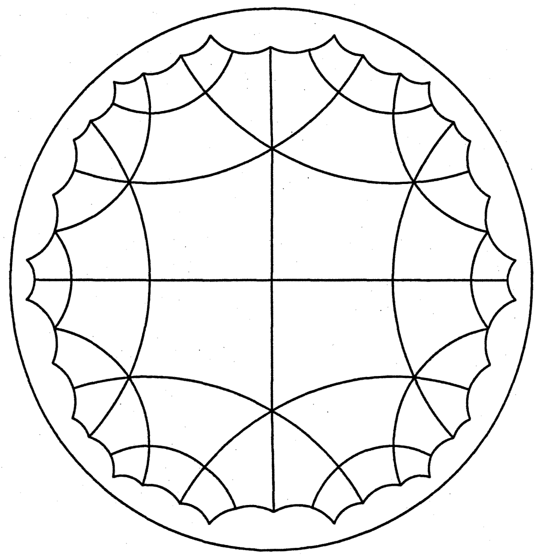
4. Kaleidoscopic Saccheri and Lambert quadrilaterals

The Saccheri quadrilaterals have two consecutive right angles on a side named base, and two equal and acute angles opposite to the base. The construction of the Saccheri quadrilateral can be solved by means of the translation of a point according a line which does not contain it.

The Lambert quadrilateral is a quadrilateral with three right angles. The union of two Lambert quadrilaterals is a Saccheri quadrilateral: that is, it has two adjacent right angles and two equal acute angles. And, conversely, if in a Saccheri quadrilateral we trace the unique perpendicular line that is common to the base and the opposite side, we obtain two Lambert quadrilaterals, congruent by means of reflection with respect to the common perpendicular.

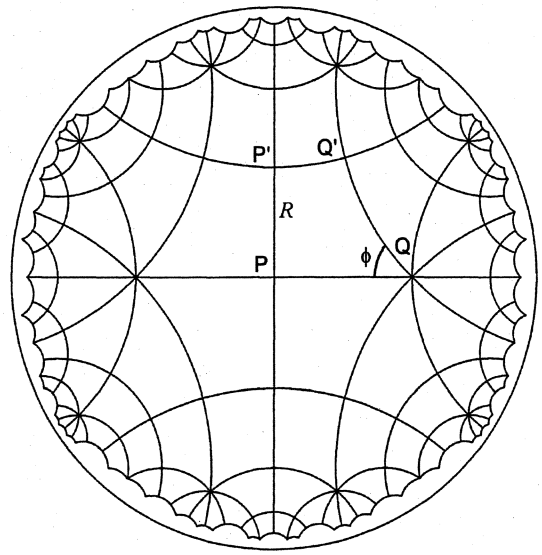
We shall now consider the determination of the quadrilaterals of Saccheri and Lambert that can tessellate the hyperbolic plane.

Theorem 3 ([7]) For every $R > 0$ and $n \in \mathbb{N}$, $n > 2$, there exists a unique Saccheri quadrilateral with the base R and acute angles $\varphi = \pi/n$, unique up to congruence, that tessellates the hyperbolic plane.

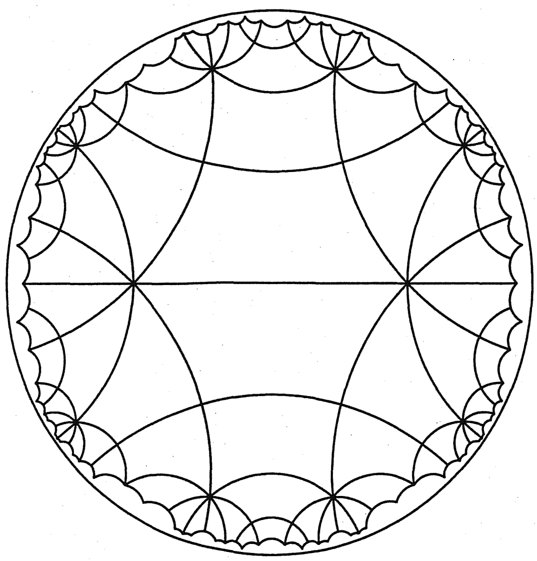
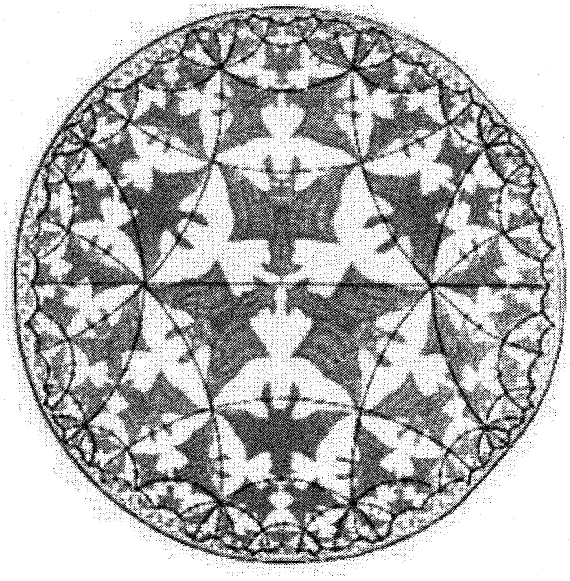


Example 2 As an example illustrating this situation, we show below tessellation of a Poincaré disc by means of a Saccheri quadrilateral for $\varphi = \dfrac{\pi}{3}$ and $R = 1$, created with the *Hyperbol* package for Mathematica software that has been developed by the authors.

Theorem 4 ([6]) For each $R > 0$ and $n \in \mathbb{N}$, $n > 2$, there exists a unique Lambert quadrilateral $\mathcal{L}(P, Q, Q', P')$ with three right angles, side P, P' or length R, and acute angle $\phi = \pi/n$, unique up to congruence, that tessellates the hyperbolic plane.



Example 3 As an example illustrating this situation, we show below two tessellations of the Poincaré disc, created also with the *Hyperbol* package, using a Lambert quadrilateral for $\phi = \pi/4$, and $R \simeq 0.881373$.

Example 4 We use the Saccheri quadrilateral that is obtained from the union of the Lambert quadrilateral, of the above example, and its reflection with respect to the line that contains the segment of length R. This Saccheri quadrilateral is subjacent in the M.C.Escher's "Circle limit IV" (*Cirkellimiet IV*) This work is based on the union of three consecutive triangles, which contain the figures appearing in this creation.

5. Tessellations in the Poincaré half-plane

All the mathematical tools that we have described for D^2 can be transcribed in the model of the Poincaré half-plane denoted by H^2. Both models are geometrically equivalent and the interested reader can find this equivalence in [3]. The *Hyperbol* package, mentioned above, also features drawing tools to work on the half-plane. Their usefulness is evident when we regard some of the works by M.C. Escher based upon this model. Two examples follows:

Example 5 Tessellation of the Poincaré half-plane, in the study work made with triangular groups ([4]), M.C.Escher's "Regular Division of the Plane V" ©2003 Cordon Art B.V.- Baarn - Holland. All rights reserved.(*Woodcut VI / Houtsnede VI*).

M.C. Escher writes in [13] his ideas over the regular division of the plane: *Before going on to discuss each of the illustrations, I should like to indicate the method used in all of them except woodcut VI to represent the different systems...* This special difference for Woodcut VI over the precedent illustrations (woodcuts I – V) is the difference between the Euclidean Geometry and the Hyperbolic Geometry.

For this woodcut, M.C. Escher writes: *Woodcut VI is the only example in this series in which the division of the plane requires more than two shades. A complete survey of the possibilities of the regular division of the plane would need to contain at last twenty illustrations...*

One of the M.C. Escher's finished works in the hyperbolic half-plane is the following:

Example 6 M.C. Escher's "Butterflies" (*Schmetterlinge*) ©2003 Cordon Art B.V.- Baarn - Holland. All rights reserved ([14]). This work is based on an easily recognizable polygonal NEC group.

6. Conclusions

We refer the reader to [3], which presents an electronic tool named *Hyperbol*([1]) whose computational support is Mathematica software. This tool consists of modules that allow us to draw different hyperbolic constructions in Poincare's models for the hyperbolic plane, usually denoted by H^2 and D^2. Such constructions include reflections, rotations, translations, glide reflections, and the orbits of a point. These isometries and geometric loci act on the hyperbolic plane; and if a euclidean element appears in some representation of this plane, we shall note it in an explicit way.

[1]Software available at http://www.ugr.es/local/ruiz/software.htm

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