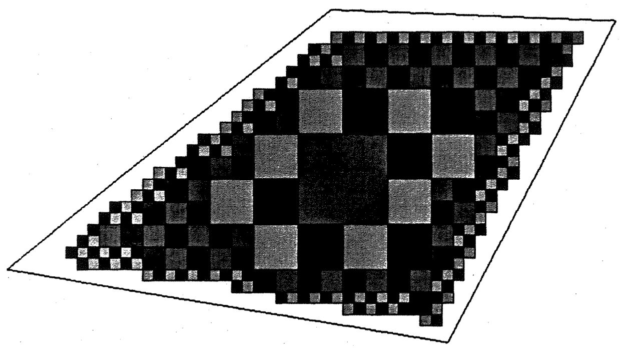
Square Decompositions with Hyperbolic Consequences in Art, Chemical Physics and Mathematics

Robert G. Smits
Department of Mathematics
New Mexico State University
Las Cruces, New Mexico 88003
Email: rsmits@emmy.nmsu.edu

Abstract

We investigate discrete versions of hyperbolic geometries which arise at interfaces within art, mathematics, physics and other disciplines. A softening from algebraic isometries to analytic inequalities gives a simple way to capture the hyperbolic spirit using Euclidean notions.



1. Introduction

Grid geometries and their transformations have a long history in art, science, music and mathematics. For example, in music the visualization of tonal space is nearly always discrete [Hoo], while in Dürer's art (noted by Sharp [Sha]) grids were used to transform faces Figure 1, and the physics of crystals as well occurs on a regular lattice.

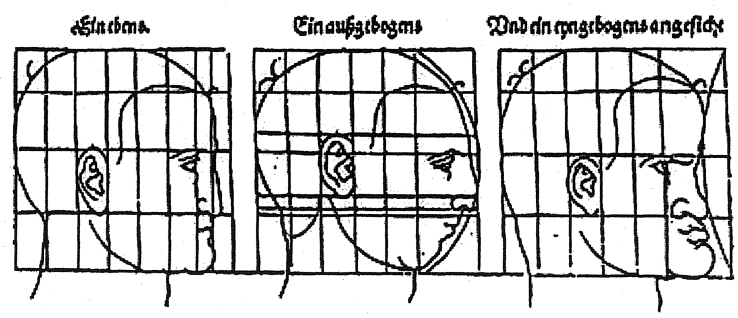


Figure 1: Dürer heads

All of these situations correspond mathematically to discrete graphs in Euclidean geometry of one, two or three dimensions which are translation invariant. The currently controversial tome "A New Kind of

Science" by noted physicist and Mathematica creator, Stephen Wolfram, actually contends that all physical events should be thought of as taking place in the discrete universe of cellular automata [Wol].

It is a general belief that whenever discrete geometries turn up they are either Euclidean or fractal in nature. While this is often the case, it is by no means exhaustive as it fails to include the rich landscape of hyperbolic phenomena. Continuous models of hyperbolic geometry occur frequently in mathematical physics and have been extensively studied. On the other hand the ideas of discrete hyperbolic geometry are more recent and still under examination.

2. Discrete Hyperbolic Geometry Outside of Mathematics

In chemistry, a polymer is basically a huge molecule of hundreds or thousands of smaller molecules in repeating units. These polymers can be found in ideal shapes as either flexible coils or rigid rods, as shown in Figure 2.

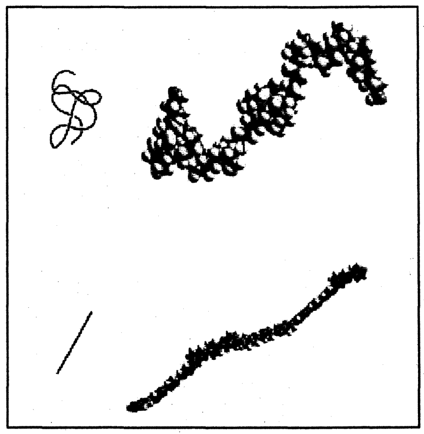


Figure 2: Polymer chains [AT]

At an interface these chain-like objects when put in a solution, adsorb to the wall and form grids whose mesh size at distance z from the wall is approximately z [deG]. Some resulting geometries and their dependence on the interfaces are illustrated in Figures 3 and 4 below.

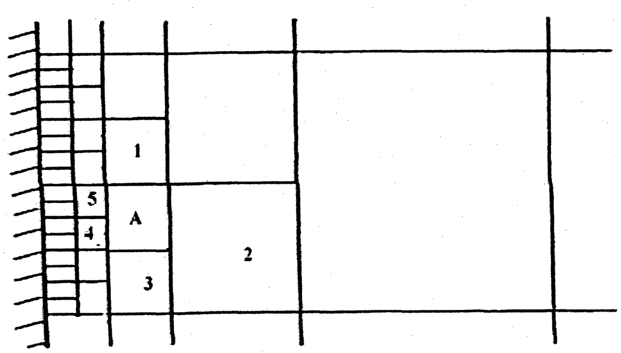


Figure 3: Absorbed polymer layer

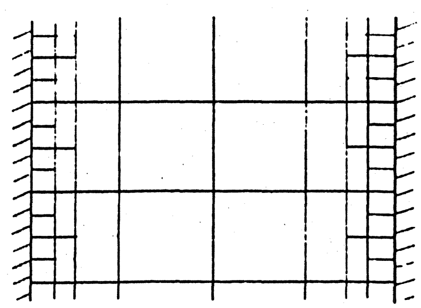
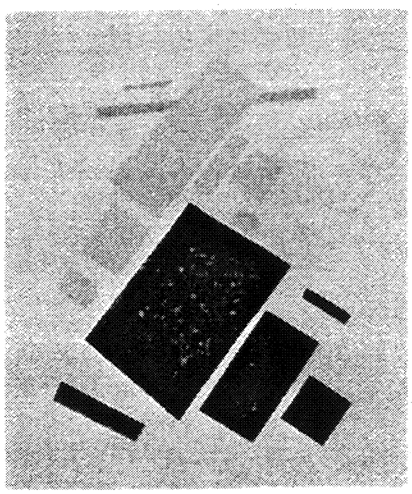


Figure 4: Reflected absorbed polymer layer

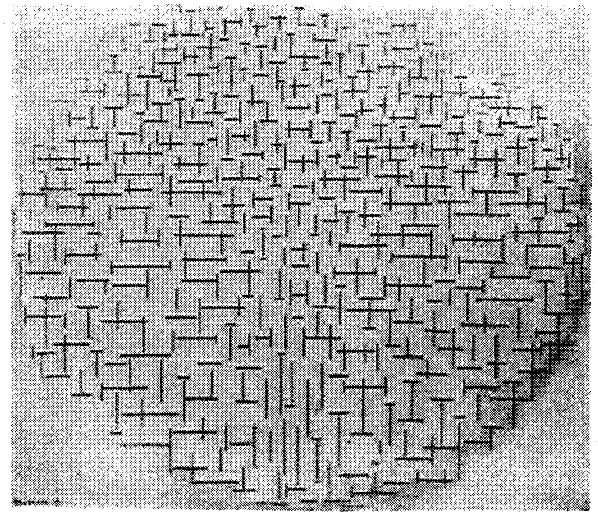
What is interesting about these structures is the growth of neighboring meshes. For example, in Figure 3 block A has 5 blocks attached, 13 blocks two away, and looking toward the boundary it is easily seen to have at least 2^n blocks within a distance of n. This exponential growth in the number of neighbors from a graph theory point of view indicates that we are dealing with a hyperbolic geometry, since in Euclidean geometry area grows polynomially. In a hyperbolic sense there is no real difference between the above figures, since both have boundaries which are not attainable using a finite number of cubes. The only difference is that Figure 4 is in a strip-like region while Figure 3 takes place in a half-plane.

Analogous geometric structures appear as well in the biology of some biofilm formations, where bacteria position themselves following a pattern similar to Figure 4.

In art, there are numerous examples of artists decomposing space with shapes whose sides are more or less proportional to their distance to boundaries. This is especially true in modern art, and can be seen in the works of abstract artists such as Kasimir Malevich or Piet Mondrian shown belo.



Malevich: Suprematist Composition 1914



Mondrian: Composition 1915

3. Rigid Hyperbolic Space in Mathematics

The invention of hyperbolic geometry, credited to both Bolyai and Lobachevski, is one of the great accomplishments of mathematical thought. The idea of this non-Euclidean geometry is so daring that many historians believe Gauss himself had deduced its existence, and kept it hidden so as not to upset the culture of the time.

A now classical approach to introducing non-Euclidean geometry and its behavior is based on the upper half-plane model of Henri Poincaré. Beginning with pairs of real numbers $\{(x,y),\ y>0\}$ as the underlying points one simply declares that the maps that will keep relative distances invariant, called *isometries*, are compositions of particular reflections in circles. Reflection in a circle, also known as inversion, is defined for a fixed circle C, with center Z and radius r to map the point P to P' in such a way that Z, P and P' are collinear, and $ZP.ZP' = r^2$, as in the figure below.

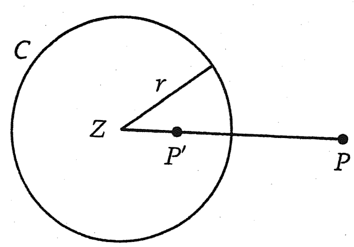


Figure 5: [Sti]

In the Poincaré model, the isometries (distance preserving maps) are built up from the reflections in circles having centers on the x-axis or reflections in vertical lines, which can themselves be thought of as reflections in circles with infinite radii. This construction captures many of the natural aspects of

Euclidean geometry. Thus, the lines (also known as geodesics) for the upper half-plane non-Euclidean geometry are the semicircles with center on the x-axis or vertical lines.

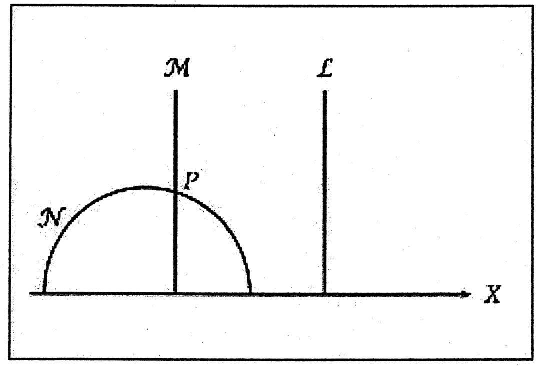


Figure 6: Some non-Euclidean lines

An example of non-Euclidean behavior is illustrated in Figure 6. Here we see two lines, M and N, passing through P neither of which intersect L, that is we have two distinct lines parallel to L, containing the point P. An important consequence of this multiplicity of parallel lines is that compositions of reflections in nonintersecting lines produce a more complex behavior than the simple translations of Euclidean geometry. In turn, this unusual behavior leads to an exponential growth of a circle's area as a function of its radius. The very modern spirit of describing geometry, hyperbolic or otherwise, as arising from compositions of actions of reflections is essentially due to Felix Klein in the 1870's [Sti]. This process approach to geometry, focusing on actions instead of elements, must fit in nicely with Leonard Shlain's physicist Feynman and artist Pollock [Shl].

Reflections in circles are also intimately related to harmonic analysis and especially complex analysis. A prime example of this connection can be seen in the double nature of the function $f(z) = \frac{1}{\bar{z}}$. The function $f(z)$ is coanalytic, and hence is a harmonic function away from z=0; at the same time it represents the reflection of a point z with respect to the circle of center (0,0) and radius 1. Complex analysis is further connected with hyperbolic geometry in that the distance preserving maps of the hyperbolic plane also preserve angles, and so it is possible to transfer the geometry of Poincaré's half-plane model to other regions using angle-preserving (conformal) maps. One classical way of applying this approach is to define the hyperbolic "plane" on the unit circle with "lines" being circular arcs perpendicular to boundary, as shown in Figure 7. Another example is given in Figure 8 showing a geodesic, constructed with the mathematical software package Matlab, for the hyperbolic geometry on a polygon via the Riemann map from the upper half-plane.

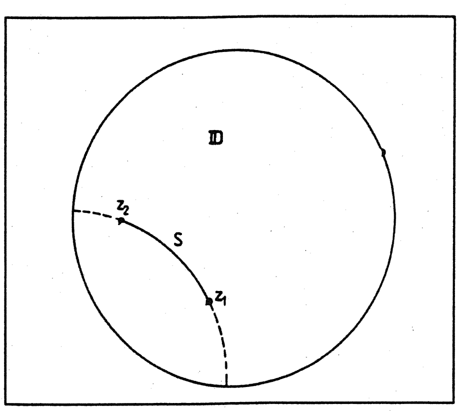


Figure 7: non-Euclidean segment S joining z_1 and z_2

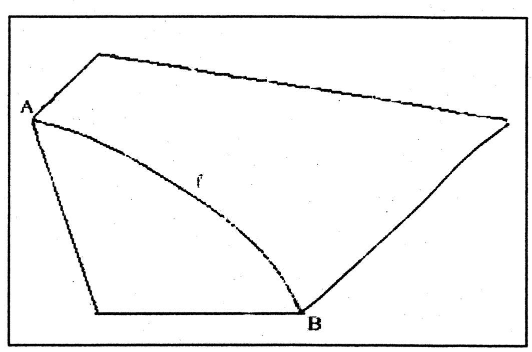


Figure 8: non-Euclidean segment joining A and B

4. Softening the Hyperbolic Plane

The insights of the Klein method are a very powerful approach to geometry, yet they are removed from our everyday experience. The tools to do hands-on measuring of quantities, such as length and volume, are often buried within an abstract setup. One way to regain the arithmetic of hyperbolic geometry is to replace continuous sets with approximating discrete sets (graphs), and isometries with *quasi-isometries*. [Coo] This current geometric viewpoint, known as *softening*, can be carried out by decomposing a region into squares satisfying two basic properties. First, the size of any square in the decomposition is comparable to its distance to the boundary, and second, adjoining cubes differ in side length by at most a factor of two. The existence of such decompositions is guaranteed by a now classical theorem of Whitney [Ste]. A distance between cubes can then be defined as the length of a chain which connects them using the fewest cubes. Paths with this chain distance are close to hyperbolic lines, as is seen in Figure 9.

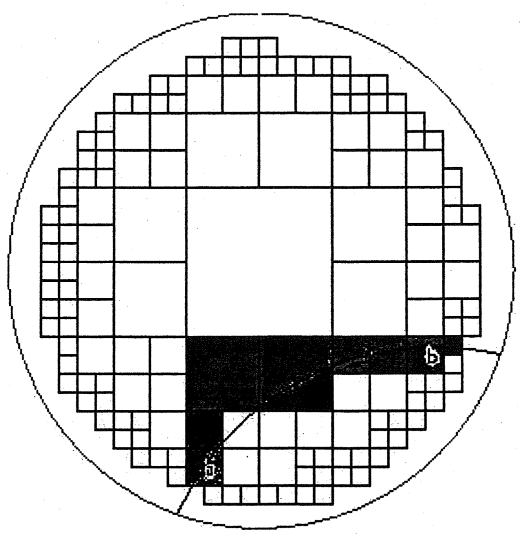


Figure 9: Cubes versus non-Euclidean from a to b

For the upper-half plane, decompositions of this type occur in many self-similar structures in physics, recall Figure 3-or in mathematical art as in Figure 10, where the boundary is represented by the bottom line.

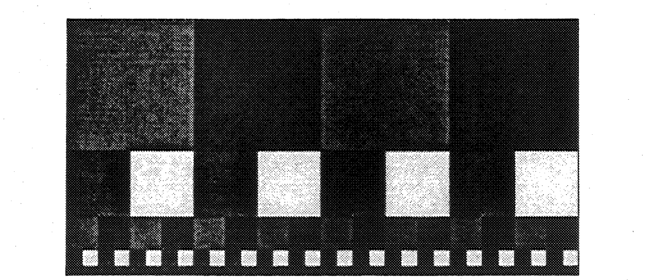


Figure 10

There are numerous tradeoffs in exchanging an equals sign in geometry, originating with Descartes' coordinates, for the inequalities of analysis and counting popularized by M. Gromov [Coo]. A square decomposition is not unique, symmetries and transformations are harder to define using inequalities, and ideals such as points and lines exist only as limits. On the other hand, many natural examples exhibit discreteness, so therefore a discrete hyperbolic geometry is useful in highlighting buried geometric information. For example looking at Figure 8, one sees the non-Euclidean line for the polygon, but two facts are hidden in here. It is unclear whether a small piece cut far away from A and B would change the line segment significantly; a second mystery is: which points are in a neighborhood of the geodesic. On the other hand with a square decomposition of the same region, as given in Figure 11, it looks evident that small changes or perturbations far away from a chain do not have a significant impact on paths, since the decomposition contributing to the chain remains unchanged. Additionally, it is also easy to see which points are near a geodesic, as the squares themselves are the geodesic.

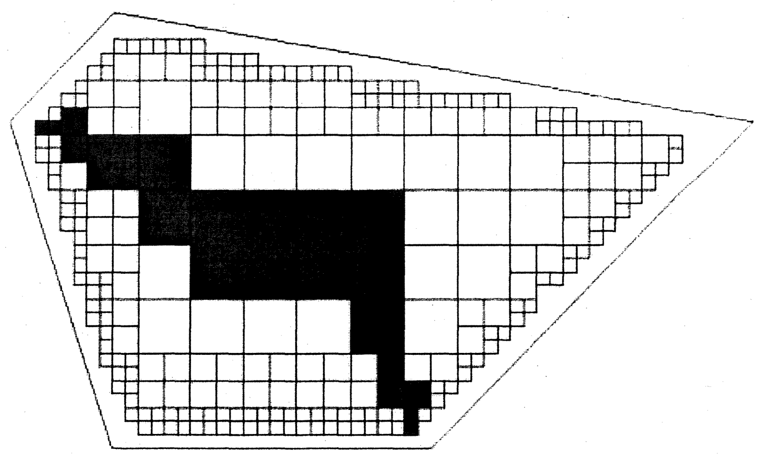


Figure 11

5. Conclusion

In this paper we explored the presence of discrete hyperbolic geometries in the arts and sciences. We underlined the existence of discrete hyperbolic structures in chemical physics and we show how these ideas are also found in art. We discussed the evolution of hyperbolic geometry, arriving at a modern approach of softening, which uses quasi-geodesics and discrete models to reveal interesting geometric information, which can be hidden by an abstract continuous model.

References

[AT] S. M. Alien, E. L. Thomas. *The Structure of Materials.* John Wiley &; Sons, Inc., New York, 1998.

[Coo] M. Coornaert, T. Delzant, A. Papadopoulos. *Géométrie et théorie des groupes* Lecture Notes in Mathematics, 1441 Springer-Verlag, 1990

[deG] P.-g. de Gennes. *Simple Views on Condensed Matter.* World Scientific Publishing Co., 1998.

[Hoo] J. Hook. *Hearing With Our Eyes: The Geometry of Tonal Space.* Bridges Proceedings 2002

[Sha] J. Sharp. A *Transformation Sketchbook.* Bridges Conference Proceedings 2002

[Shl] L. Shlain. *Art and Physics: Parallel Visions in Space, Time and Light.* Bridges Proceedings 2002

[Ste] E. Stein. *Singular Integrals and Differentiability Properties.* Princeton University Press, 1970

[Sti] J. Stillwell. *Numbers and Geometry.* Springer Verlag 1998

[Wol] S. Wolfram. *A New Kind of Science.* Wolfram Media, Inc., 2002