

The Art in Polynomiography of Special Polynomials

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Abstract

Polynomiography is the art and science of visualization in approximation of zeros of complex polynomials. It allows one to obtain many colorful images of polynomials. These images can subsequently be re-colored in many ways to create artwork. Polynomiography has tremendous applications in the visual arts, education, and science. Artistically, it can be used to create diverse set of beautiful images reminiscent of the intricate designs and modern art. Educationally, polynomiography can be used to teach mathematical concepts. Scientifically, it provides a tool, not only for viewing polynomials, present in virtually every branch of science, but also a tool to discover new theorems. The goal of this paper is to present some artwork produced via polynomiography of a few polynomials arising in science as well as a few considered to arrive at beautiful but anticipated designs. These include a Chebyshev polynomial, a polynomial arising in physics, one in knot theory, and some based on roots of unity. The purpose of the paper is doing art on these special polynomials. But the reader will realize that these images also help engrave certain attributes of these polynomials. Moreover the beauty of these images suggests an infinite jewel box of polynomials yet to be discovered and visualized through polynomiography by future polynomiographers. The ultimate goal here is to suggest that polynomiography is indeed a new art form that can be thought of as "painting by numbers" or "painting by points." In a sense it is one of the most minimalistic art forms.

1. Polynomiography

1.1. Introduction.

Polynomiography is defined to be "the art and science of visualization in approximation of zeros of complex polynomials, via fractal and non-fractal images created using the mathematical convergence properties of an infinite family of iteration functions." An individual image is called a "polynomiograph." These images are obtained using variety of algorithms. In this paper I present some artwork based on homemade prototype polynomiography software. Polynomials form a fundamental class of functions that arise in virtually every branch of science. Polynomiography has many facets and could appeal to many. As an artistic tool everyone can use it: children, middle school or high school students, and the general public. Using polynomiography software could be no more complicated than using an ordinary camera: we choose a subject to photograph, adjust the settings, and then shoot a picture. Except that in polynomiography we shoot pictures of polynomial equations, and the camera is the polynomiography software. But we also have the ability to zoom in or re-color the initial polynomiographs employing our own creativity and imagination using the polynomiography software. The range and diversity of images obtained through polynomiography is indeed quite extraordinary and surprising.

Polynomiography is not the only technology for creating computer-generated art. It does for instance overlap with fractals in the sense that, according to the new definition, some previously given fractal images may now be called polynomiographs. But polynomiographs are not necessarily fractal images.

The word “fractal” coined by the Benoit Mandelbrot refers to sets or geometric objects that are self-similar and independent of scale. Some fractal images can be obtained via simple iterative schemes leading to sets known as Julia set and the famous Mandelbrot set. The simplicity in the creation of such images has resulted in numerous web sites where amateurs and experts exhibit their fractal images. The fact that a polynomiograph may not be fractal is one of the reasons why it was necessary to define a new name. A second reason is to emphasize its objective and means, i.e. visualization of polynomials via approximation of its roots and the way in which the approximation is carried out. Even when a polynomiograph is a fractal image it does not diminish its uniqueness. Indeed even within the fractal images derived from polynomiography there lies more variety, diversity, control, creativity, as well as the element of surprise than typical fractal images based on mere iterations. Polynomiography makes it possible to consider a new problem that I call, the “Reverse Root-Finding”: given a set of points find an iteration function that will result in a desirable polynomiograph. The polynomiographs given in Figure 1 and Figure 2 are such examples.

The foundation of polynomiography is based on a well-defined goal: visualization in approximation of zeros of polynomials. Using various properties of convergence, as well as variety of coloration schemes, it provides two-dimensional images of the process of approximation, viewed through the lenses of an infinite family of iteration functions. An iteration function is a rule that assigns to each given point in the Euclidean plane another point in the plane. Starting with a given point as input an iteration function repeatedly generates a new output, which is fed back to the iteration function as a new input. Over a period of time one can trace the history of the initial input as it moves from one location to another. Polynomiography makes use of different schemes to color a point based on its history. Leaving aside the artistic aspects in polynomiography, I remain convinced that it will bring new insights into the ancient problem of root-finding, even for the experts in the field of root-finding.

A polynomial maybe completely characterized by a finite collection of points in the Euclidean plane. These points are its “roots” or “zeros.” Each root “rules” a certain “territory.” Thus the roots divide the Euclidean plane into regions of their own territories. The rule according to which the territories are determined is carried out according to an iteration function. And there are infinitely many iteration functions. Since a polynomial is characterized by its roots or its coefficients, polynomiography can be viewed as “painting by points,” or “painting by numbers.” In this sense it is one of the most minimalistic ways to create artwork. But even as many as ten, twenty or thirty points can produce such fantastically complex, yet diverse set of images that no human being could produce in a life time

A general *complex polynomial* of degree n is an expression of the form

$$p(z) = a_n z^n + a_{n-1} z^{n-1} + \dots + a_1 z + a_0$$

where n is a natural number, or zero, and a_n, \dots, a_0 are complex numbers, its *coefficients*. According to the Fundamental Theorem of Algebra, a polynomial of degree n , with real or complex coefficients, has n real or complex zeros (roots) that may or may not be distinct. The problem of approximation of zeros of polynomials, considered even by the Sumerians (third millennium B.C.), has been one of the most influential problems in the development of several important areas in mathematics (see [22] for a survey). Polynomiography is a new approach to solve and view this ancient problem, while making use of new algorithms and today's computer technology. Polynomiography is based on the use of one or an infinite number of iteration functions designed for the purpose of approximation of roots of polynomials. An iteration function is a mapping of the plane into itself and can be viewed as a machine that approximates a zero of a polynomial by an iterative process that takes an input and from it creates an output that in turn becomes a new input to the same machine.

2. A Mathematical Foundation of Polynomiography

To do polynomiography we need to select a polynomial and then an iteration function. The best-known iteration function for finding the roots of a polynomial is the Newton iteration function. But this is only one iteration function and hence very limited. What I have used as the main source of iteration functions is a fundamental family of iteration functions called the "Basic Family," represented as $\{B_m(z)\}_{m=2}^{\infty}$. The algebraic development and some optimal properties of the Basic Family are described in [7] and [8]. To describe the formula for the members of the Basic Family consider $p(z)$ a polynomial of degree $n \geq 1$ with complex coefficients. Set $D_0(z) \equiv 1$, and for each natural number $m \geq 1$, define

$$D_m(z) = \det \begin{pmatrix} p'(z) & \frac{p''(z)}{2!} & \cdots & \frac{p^{(m-1)}(z)}{(m-1)!} & \frac{p^{(m)}(z)}{m!} \\ p(z) & p'(z) & \ddots & \ddots & \frac{p^{(m-1)}(z)}{(m-1)!} \\ 0 & p(z) & \ddots & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & \frac{p''(z)}{2!} \\ 0 & 0 & \cdots & p(z) & p'(z) \end{pmatrix},$$

where $\det(\cdot)$ represents determinant. For each $m \geq 2$, define

$$B_m(z) \equiv z - p(z) \frac{D_{m-2}(z)}{D_{m-1}(z)}.$$

The first member of the Basic Family, $B_2(z)$, is Newton's function, and $B_3(z)$ is Halley's function. For the history of these two iteration functions see [22], [23]. It is also possible to describe the formula recursively. The Basic Family has numerous fundamental properties. It is closely related to a nontrivial determinantal generalization of Taylor's theorem, see [10]. For the multipoint version of the family and their order of convergence see [10] and [9], respectively. For a detailed list of theoretical and computational properties see [9]-[17]. In [16] I exhibit many images and give more detailed description of potential applications of polynomiography. There are two basic but very important properties of the Basic Family. The first one is that we can select any natural number m greater than or equal to 2 and then apply the fixed-point iteration

$$a_{k+1} = B_m(a_k), \quad k \geq 0$$

where a_0 is a starting complex number. When the starting point is close to a simple root of the polynomial the fixed-point iterates converges to the root having order m (i.e. in each iteration the number of correct digits, approximately, gets multiplied by m). The higher the order of convergence the fewer the number of iterations needed to locally approximate a root. Having order of convergence equal to 2 or 3 may not make much of difference locally, but globally, and certainly in terms of polynomiography the difference between order two and order three methods is analogous to the difference between using two completely different lenses to do photography. Using the iteration function $B_m(z)$ we can generate the sequence of fixed-point iterates for each point within any given rectangular region and trace its history. In practice we will divide the rectangular region into pixels so that we will deal with a finite number of points. We then apply various color-coding to these pixels to get an image. Each iteration function works like a lens. Thus the more iteration functions, the more lenses we have for viewing polynomials. But there are other benefits in using families of iteration functions and particularly, the Basic Family.

To describe another fundamental property of the Basic Family, consider the set of distinct roots of $p(z)$. The elements of this set partition the Euclidean plane into Voronoi regions of the roots and their boundaries. The Voronoi region of a root is a convex polygon defined by the locus of points that are closer to this root than to any other root. The boundary of the Voronoi regions is the locus of points that are equidistant from two distinct roots. This is a set of measure zero consisting of the union of finite number of lines. Given a complex number a , the *Basic Sequence* at a is defined as $\{B_m(a), m = 2, 3, \dots\}$. Now given $p(z)$, for any input a not in the boundary of the Voronoi regions, the Basic Sequence is well defined and converges to some root. It can also be shown that the root to which the Basic Sequence converges is in fact the closest root to the given input. This gives another scheme for coloring.

2.1. Basins of Attractions and Voronoi Region of Polynomial Roots

The basins of attraction of a root of $p(z)$ with respect to the iteration function $B_m(z)$ are regions in the complex plane such that given an initial point within them the corresponding sequence of fixed-point iterates will converge to that root. The boundary of the basins of attractions of any of the polynomial roots is the same set. This boundary is known as the Julia set and its complement is known as the Fatou set. The fractal nature of Julia sets and the images of the basins of attractions of Newton's method are now quite familiar for some special polynomials, e.g. $p(z) = z^3 - 1$ (see [5]). Mandelbrot's work (see [18]) in particular popularized the Julia theory [6] on the iteration of rational complex functions, as well as the work of Fatou [4], and led to the famous set that bears Mandelbrot's name. Peitgen et al. [20] undertake a further analysis of fractals. Mathematical analysis of complex iterations may be found in Peitgen and Richter [19], Devaney [2], and Falconer [3].

Although the fractal nature of the Julia sets corresponding to the individual members of the Basic Family follows from the Julia theory on rational iteration function, that theory does not predict the total behavior of specific iteration functions on the complex plane. In particular, the behavior of the Basic Family, individual members or collectively is not a consequence of Julia set theory.

3. Polynomiography of Some Special Polynomials

In this section I will present some polynomiographs corresponding to a few polynomials. The purpose of this section is to only give a flavor of the power and diversity of polynomiography. For more images and their surprising diversity I refer the reader to my web site www.polynomiography.com. In particular, [16] describe polynomiography in more detail and offers many more of my artwork

3.1. Polynomiography of Roots of Unity. The first classes of simple polynomials that come to mind are those based on the roots of unity, i.e. solutions to $p(z) = z^n - 1$. If we chose n to be 20 we expect to get twenty roots equally spaced on the unit circle. If we now consider $p(z) = (2z)^n - 1$ then the corresponding roots will be at a circle of radius half. And if we multiply the roots of unity for $n=20$, and the roots of the latter with $n=10$, then we will arrive at a polynomial with 30 roots. The following figure shows some polynomiographs of this very simple polynomial. The first two images are created using properties of the Basic Sequence, while the right-most image makes use of a particular member of the Basic Family and coloring based on of the fixed-point iterations.

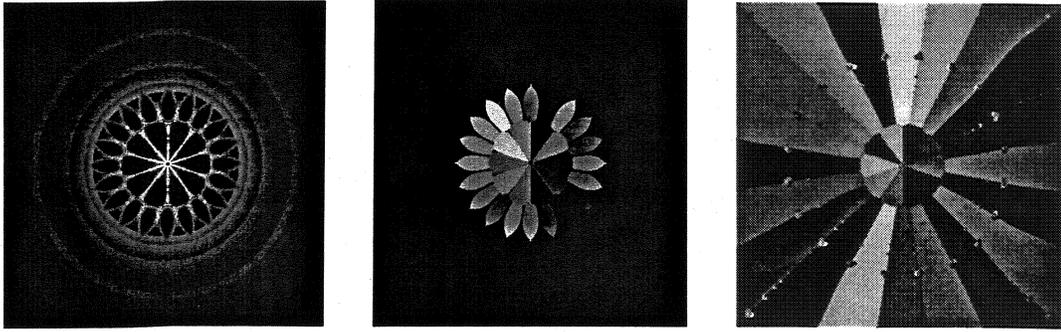


Figure 1: Polynomiographs of the product $[(2z)^{10} - 1] \times (z^{20} - 1)$.

The following polynomiograph was inspired by a Persian carpet and was produced using roots of unity and their rotations. The image in turn is being turned into a carpet two meters in diameter.

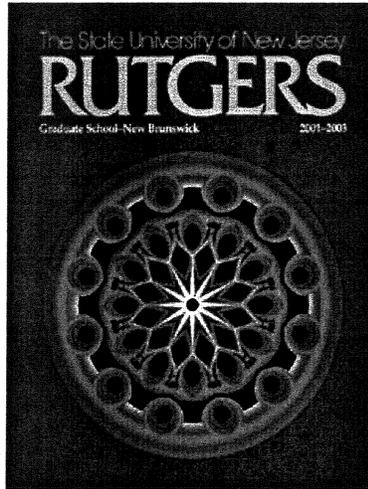


Figure 2: Polynomiograph of a degree 36 polynomial.

The next polynomiograph is one is obtained from a high degree root of unity using the Basic Sequence.

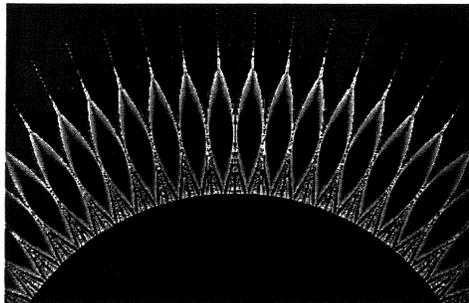


Figure 3: "Statue of Liberty."

3.2. Polynomiography of a Chebyshev Polynomial. Polynomiographs here are those of a fifth degree Chebyshev polynomial using either the properties of Basic Sequence or a particular member of the Basic Family.

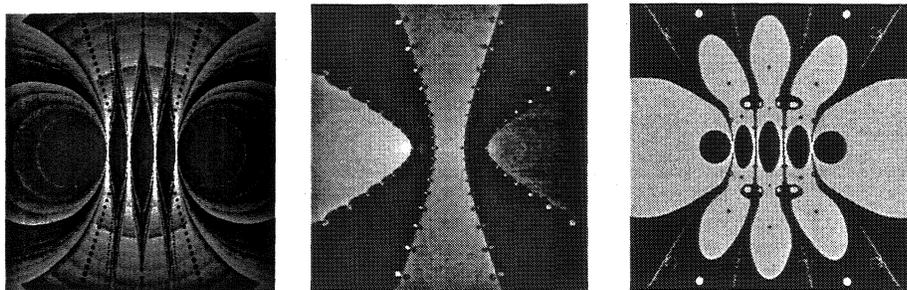


Figure 4: Polynomiographs of $T(z) = 16z^5 - 20z^3 + 5z$.

3.3. Polynomiography of a Polynomial in Physics. The next set of images come from a polynomial arising in physics. Nature has its own beautiful polynomials.

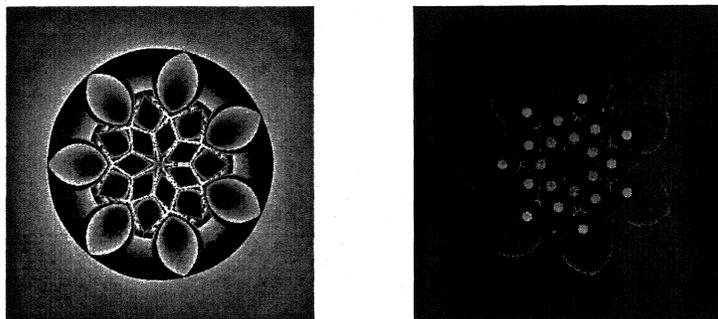


Figure 5: Polynomiographs of a Physicists' polynomial $z^{21} + 5z^{14} - 22z^7 / 15 - 11 / 675$.

3.4. Polynomiography with a Polynomial in Knot Theory. The next two images are based on Alexander polynomial arising in knot theory. The original images are rotated. The "baby" image was subsequently re-colored to give the desired effect (the hands).

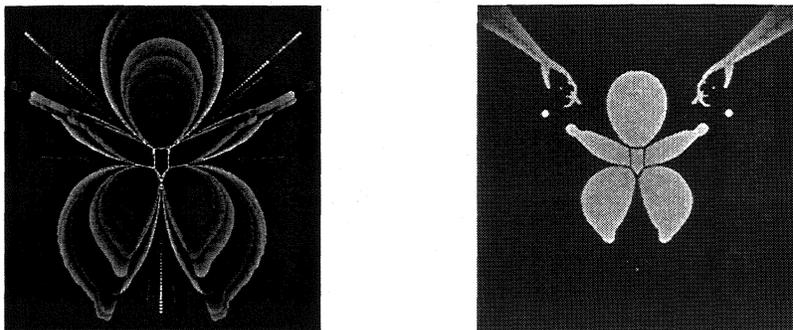


Figure 6: "Butterfly", and "Handling a Baby" from $z^6 - 3z^5 + 4z^4 - 5z^3 + 4z^2 - 3z + 1$.

4. Concluding Remarks and Future Work

I believe that through various software programs not only polynomiography could grow into a new art form for both professional and non-professional art making, but also as a tool with enormous applications in the teaching of art and mathematics. While a polynomiography software can be used to teach both art and mathematics more effectively, another can be used for the visualization of polynomial properties by an advanced researcher. Polynomiography software could be used in the mathematics classroom as a device for understanding polynomials as well as in the visualization of theorems pertaining to polynomials. As an example, high school students studying algebra and geometry could be introduced to mathematics through creating designs from polynomials. They would learn to use algorithms on a sophisticated level and to understand mathematics of polynomials in its relationship to pattern and design in ways that cannot be approached abstractly. At a higher educational level, e.g. calculus or numerical analysis courses, polynomiography allows students to tackle important conceptual issues and gives the student the ability to understand modern discoveries such as fractals. Explorations of artistic, educational and scientific applications of polynomiography are among my primary, present and future, plans. In particular, in the future I will introduce software and books on the subject of polynomiography. The interested reader may visit www.polynomiography.com for more information. The web site also includes a Java version of polynomiography software where the visitor can produce his/her own images and get a flavor of how the software works. The software however is a limited version of the PC version of a homemade polynomiography software.

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