BRIDGES
Mathematical Connections in Art, Music, and Science

Musical Acoustics and Continued Fractions

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Equal-tempered systems (ETS) "close the octave" by forcing all frequency ratios to be the same for adjacent notes. In our familiar 12-tone equal-tempered system, the frequency ratio of adjacent notes is $2^{1/12}$. Therefore, the octave is reproduced every twelve notes. ETS make transposition and modulation easy but compromise the consonant/harmonious ("just") intervals. The generation of equal-tempered musical scales that best approximate just intervals has a continued fraction basis[1, 2, 3]. Suppose we want to generate an ETS that best approximates the perfect fifth and closes the octave. In other words, we want to find integers p and q so that $2^{p/q} \approx 3/2$ or $|p/q - \log_2(3/2)| \approx 0$. Because we want to approximate an irrational number by a rational fraction the continued fraction approximation is the natural choice. Therefore, we expand $\log_2(3/2)$ as a continued fraction to find p/q. Douthett, Entringer, and Mullhaupt[4] have shown that the principal continued fraction convergents "maximize the harmonious advantages" of an ETS by most closely approximating the target just interval (the perfect fifth in this example). The denominators of these continued fractions represent the number of notes needed in the octave of an equal-tempered system. The number of equal-tempered steps that best approximates the target interval is given by the numerator of the continued fraction. In this presentation we will demonstrate how other equal-tempered musical scales are generated using continued fractions and show how continued fractions may be used to analyze non-standard tunings.

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[1] M. Schechter, "Tempered Scales and Continued Fractions," Am. Math. Monthly, **87**, 40-42 (1980).
[2] R. J. Krantz, "Continued Fraction Compromise in Musical Acoustics," Am. J. Physics, **66**(4), 276-277 (1998).
[3] J. Douthett, "The Theory of Maximally and Minimally Even Sets, the One-Dimensional Antiferromagnetic Ising Model, and the Continued Fraction Compromise of Musical Scales," Ph. D. dissertation (University of New Mexico, Albuquerque, NM, May 1999).
[4] J. Douthett, R. Entringer, and A. Mullhaupt, "Musical Scale Construction: The Continued Fraction Compromise," Utilitas Mathematica, **42**, 97-113 (1992).