BRIDGES
Mathematical Connections
in Art, Music, and Science

The Rubik's-Cube Design Problem

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Abstract

The design problem spans art and mathematics. Its subproblems make incursions into science. We begin by stating, what the problem is -- a set of algorithms that enable us to create a three-dimensional, composite, pleasant, geometrical design on a set of Rubik's cubes. We explain parity pairs, which influence design symmetry. We actually construct a simple design, the Menger Sponge. We conclude this article with some observations about fractals. The nature of the Rubik's cube makes it possible to create designs which mimic fractals.

1. The Name of the Game

1.1 Introduction. The goal of the design problem is to construct, by conventional cube manipulation, a composite, pleasant, geometrical design on a set of Rubik's cubes. Art enters the problem via these designs, which, if I may say so myself, are not unattractive. The math is brought into the problem by the cubes themselves, and by an acute necessity to develop certain algorithms based on mathematics. The idea of using the Rubik's cube as an art medium occurred to others on the World Wide Web. The major difference is that they create two dimensional, picture-like structures, whereas my designs are fully three dimensional, perhaps similar to sculptures.

1.2 The Basics. A solution algorithm for a Rubik's cube is independent of colors on its faces. This fact led David Singmaster, an English mathematician and a leading authority on the Rubik's cube, to devise a notation for its faces and rotations. He labeled faces according to their position on a fixed cube as F for front face, B for back face, U for up face, D for down face, L for left face and R for right face. A rotation of the up face, say, is labeled as U for a clockwise rotation by 90 degrees, U2 for a rotation by 180 degrees, and U' for a clockwise (counterclockwise) rotation by 270 (90) degrees. One may consult a number of books, e. g. [1], and [2].

There are three kinds of Rubik's cubes in a given three-dimensional design. The corner cubes form corners of a design and display three faces. For a rectangular-solid or cubical design there are always eight corners. The smallest object is made up of the corners only as a 2x2x2 larger design. The edge cubes are cubes, which form edges of a design and display two faces. Finally, the center cubes are cubes that form centers of a design and display just one face. My fellow artists, who create picture designs, work essentially with center cubes.

To summarize, the basic, unrefined, and fairly simple-sounding design algorithm is:
1. Construct patterns on individual cubes
2. Color-synchronize all cubes, so that they display some geometrical symmetry
3. Stack the cubes together in the manner of a three-dimensional jigsaw puzzle; the patterned cubes, created using the previous two steps, are the jigsaw pieces.

When refining the above steps, the mathematics of the Rubik's cube and the need to display geometrical symmetry on each and every face of the design, including the bottom face confront one. I tried to simplify the first step somewhat by breaking each pattern into a set of sequences. One needs a few sequences to construct quite a lot of patterns. Instead of memorizing each pattern one merely needs to know how individual pieces of the Rubik's cube (called cubies) transfer under the action of a given sequence. This method is described in Reference [3]. A color scheme is a fixed array of colors on opposite faces of a Rubik's cube. Once a color scheme is chosen, it remains the same throughout the design. All corner and edge cubes should have identical color scheme. For the center cubes it is sufficient to have the required colors somewhere on the cube.

1.3 Parity Pairs. Parity pairs play a major role in the design problem. Without them, the designs could not be of fewer than six colors and the design symmetry, as we know it, would not exist. Suppose we have two cubes of identical color schemes, such that the colors on the U, D, F and B face of one cube are identical to the colors on the U, D, F and B face of the other cube. If the color of the L/R face of one cube is identical to the color of the R/L face of the other cube, such a pair of cubes form a parity pair, as seen in Fig. 1 (a). The cubes in Fig. 1 (b) do not form a parity pair.

Let eight cubes form four parity pairs. Place two members of such a pair next to each other. Because a pair of opposite faces of one cube is switched relative to the other cube, the two internal faces of this 2-cube structure that touch, are colored the same, leaving only five colors on its six faces. We can form four such cube structures - one for each parity pair. Those four structures can be further combined to form two 4-cube structures; each has four colors on its combined six faces. They are the top and bottom layers of the 2x2x2 clean "design." Combining them will produce the 2x2x2 clean "design" of three colors only on its six faces. To see this, please obtain eight Rubik's cubes in four parity pairs and create this "design."

Such arrangement of cubes is used as corners in larger designs, leading to reflection invariant designs. Those are (usually) cubes that display the same design on their combined opposite faces. Reflection invariance is the simplest design symmetry induced by parity pairs. In Figures 2 and 3 we see two original designs with additional parity-pair requirements. The three-color Vasarely design, shown in Figure 4, could not be constructed without a judicious use of these pairs. Space limitations prohibit further discussion of this topic.

2. The Menger Sponge

2.1 Introducing the Sponge. The Menger Sponge, shown in Fig. 5, is not an original design, but is an adaptation of a classical fractal. The centers are clean. The color of the centers must be identical to the color of the center cubes of the adjoining edges and corners. Execute the F B' U D' R L' F B' sequence on suitably oriented Rubik's cubes and, if necessary, repeat it twice. The fact that we have created this fractal from Rubik's cubes leads to the following two questions:

1) Can other fractals be created from Rubik's cubes as designs?
2) Can we learn something new about fractals from the design problem?

The answer to the first question is yes. Two other fractals are shown in Figs. 6a and 6b. The answer to the second question is that I am going to try in the next section.

3. Fractal geometry

3.1 Fractal Properties. Fractal objects are of considerable interest. There is a lot of literature on fractals on the Web. But besides showing pretty computer-generated graphics, there is little else. No real mathematics surrounds fractals. Yet I believe time is ripe for such exploration. The mathematics developed over centuries deals with stable or nearly stable systems. To complement our understanding of natural phenomena we need to consider systems that are characterized by fractals.

Fractals have two main features, fractional dimensions and self-similarity. If one constructs a smaller and smaller version of a fractal, the smaller version bears a strong resemblance to the original fractal. This property is known as self-similarity. The notion of dimension is described in Reference [4]. The formula arrived at is:

$$D = \log\text{(number of pieces)}/\log\text{(magnification)}$$

where "number of pieces" is the number of constituent pieces into which an object has been subdivided. We divide the object into pieces which, when magnified by a certain factor called "magnification," give the original object.

3.2 Fractals and Rubik's-cube Designs. Some of the designs closely resemble fractals (the Menger Sponge). A real Menger Sponge [5] has holes, which we cannot drill into the Rubik's cubes without ruining them. We pretend the holes are there by defining a "hole color" and the "background color." For the Menger Sponge the color occupied by holes is the hole color, while the surrounding pieces constitute the background color. A gasket is a "solid" fractal. A carpet is a "flat" fractal. All fractals, shown in the figures, are gaskets. A corresponding carpet is readily obtained as a picture design. The zeroth iteration is the clean cube. The cube with a suitable pattern on it is the first iteration, or seed. Define a general rule of self-similar iteration. To go from nth iteration to n + 1st iteration, we do the following:

1) If the n-th iteration cubie has the color of the background, replace it by the seed.
2) If the n-th iteration cubie has the color of the hole, replace it by a clean cube having the hole color. In other words, "enlarge the hole."

This rule has self-similarity built into it. Let us apply it to the case of a Menger Sponge. To go from first to second iteration, we examine the cubies on the seed. We replace the surrounding cubies by the Rubik's cube with seed pattern on them. The center cubie is a hole so we replace it by the clean cube. The Menger Sponge Design, shown in Fig. 5, is the second iteration. To go from second to third iteration we proceed in exactly the same manner. The seeds will replace all background-color cubies. Nine Rubik's cubes replace nine cubies of the center cube. The center cubies of the surrounding eight cubes will be replaced by clean Rubik's cubes.

3.3 The Real Set. Just as rational and irrational numbers combine to give real numbers, one should combine a dimension of fractals and spaces with integer dimension. Call this combined set a real set. A unique dimension, which is a real number, characterizes each member of the real set. One does not think of ordinary spaces as self-similar, but they are. Let us apply a rule of self-similar iteration sketched above. For a point, we invert the seed for the Menger Sponge. Let the centerpiece be background and the surrounding 8 pieces be holes. Then the dimension is $\log(1)/\log(3) = 0/\log(3) = 0$. For the line let us choose the seed to be a Rubik's cube and do U2 D2 on it. The background color is taken by the cubies in the middle layer, while the holes are occupied by the colors in the U and D layer. Using this seed we get $\log(3)/\log(3) = 1$. A second iterartion would get nine cubies in the middle layer, but the magnification also increases to 9, yielding $\log(9)/\log(9) = 1$. By placing integer-dimension spaces and fractals into a real set, we can utilize what we know about real spaces to learn about fractals and vice versa. Fractals are not esoteric objects, but merely a part of the whole picture.

3.4 Fractal Processes. The box fractal shown in Ref. [4], p. 132, is a carpet. The seed is obtained on a face of a Rubik's cube by doing, say, L2 R2 F2 B2 U2 D2 to obtain a checkerboard pattern. By choosing the corner cubies and a center cubie as background colors, and the edge cubies as hole colors, we obtain the seed of the box fractal. Using the above rule for iteration (replace background cubies by seeds and hole cubies by clean cubes) we obtain the second iteration as shown in Ref. [4]. A box fractal gasket is shown in Fig. 6a. It is called box fractal A therein. The dimension is log(5)/log(3) for a box fractal A carpet. A second box fractal in Fig. 6b is called box fractal B. Its seed is obtained by switching the hole colors and the background colors of the box fractal A. The second iteration of this fractal gasket is shown in Fig. 6b. The dimension of the box fractal B carpet is log(4)/log(3).

Instead of iterating, we combine the second iterations of both box fractal A and box fractal B to produce a third fractal, the so-called checkerboard pattern (Fig. 7). The corresponding carpet would be a fractal of dimension log(41)/log(9). This is a seed of the combined fractal, or its first iteration. The second iteration would be obtained the usual way, by taking each cubie of the background color and replacing it by the seed. Not by a single Rubik's cube, but by the whole 9-cube structure. Nine clean Rubik's cubes should accordingly replace each hole cubie. To sum up: a fractal process is a way to combine two simple fractals. The result of the combination is a third fractal which is self-similar and which has different fractional dimension. A second iteration could readily be constructed, either as a carpet or gasket, but would require 6561 and 531441 cubes, respectively! A computer programmed to take care of such iterations best carries out the second iteration. I myself would be curious to see the result.

What we will probably see is this: the lower the dimension, the less background there will be, and therefore less stability. On the other hand, the spaces with integer dimensions, too, have more stability with increasing dimensions. This is a pretty intuitive statement, but it stands to reason that a point will have a much smaller region of stability than a volume. We can combine these two observations and make the following statement: the lower the dimension (both integer and fractal) of a system is, the less stable that system will be. So, if we want to improve the stability of a system, we should strive for fractal processes whose dimension is as close as possible to an integer dimension, and translate those fractal processes to physical processes through, perhaps, the use of Mandelbrot sets. The case I have illustrated here is an idealized case, perhaps not suitable to any real physical system. But the possibility of doing that exists. One needs to utilize the computer and investigate these fractal processes.

I do not wish to imply that investigating fractal processes will be easy. This should probably be a task for the next millennium. Fractals occur around us in nature. How did they get there? Through some fractal process? Or through some process that may not be fractal? Do such processes exist? And what is the mathematics to handle them?

4. Conclusion

4.1 The Last Few Words. The design problem is multidisciplinary. It has a bit of everything: art, mathematics and science. Other subproblems to consider are to develop an algorithm and computerize the design problem for others to do. Some expert programmers should implement their skills and code the pertinent algorithms. Rubik's cube is sometimes cited as an example of cellular three-dimensional automata. Perhaps those problems can be studied with such a code.

Multidisciplinary problems shoud receive more attention in cutting-edge research. After all, Nature does not compartmentalize itself, man has compartmentalized Nature. Thus separate disciplines like physics, chemistry, biology, etc, etc, have eolved over the centuries, with their own specific rules. An attempt to cross those boundaries has been made with combination of fields, such as physical chemistry, biophysics, biochemistry and others. This attempt should be further widened. Study of fractal systems, for example, should cover many fields. Experts in those fields should benefit.

5. Acknowledgments

Thanks are due to Zdislav V. Kovarik, professor of mathematics at McMaster University, for telling me about this forum and giving me the pertinent web site. Ms. Bonnie S. Cady helped me with the formatting and introduced me to the intricacies of MS Word and picture handling. Mr. Josef Jelinek put together http://cube.misto.cz, a URL containing scanned photographs of 12 designs. And last, but not least, I would like to thank those who visited this URL. They kept the faith and kept me going. ☺

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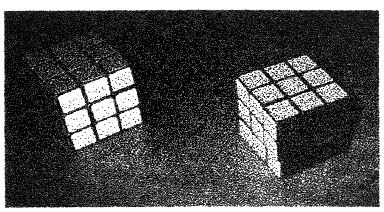

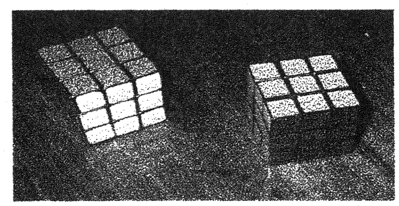

Figure 1: (a) These cubes form a parity pair (b) These cubes do not form a parity pair.

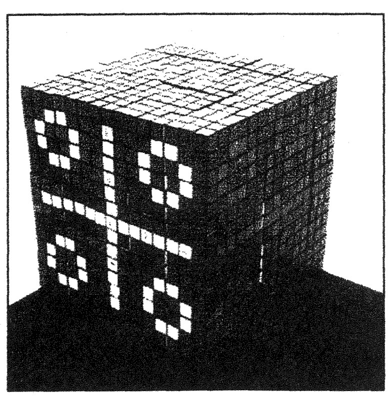

Figure 2: Marie Design

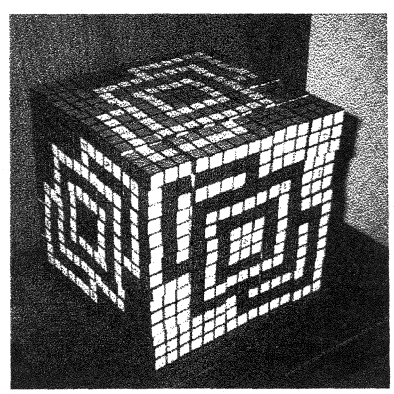

Figure 3: Jaroslav Design

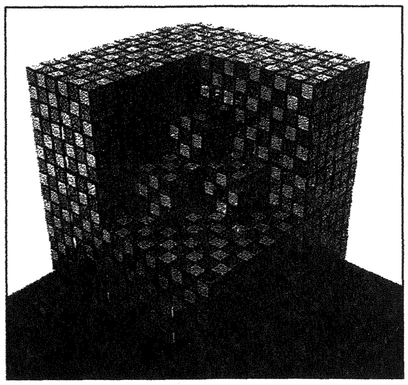

Figure 4. Vasarely Design

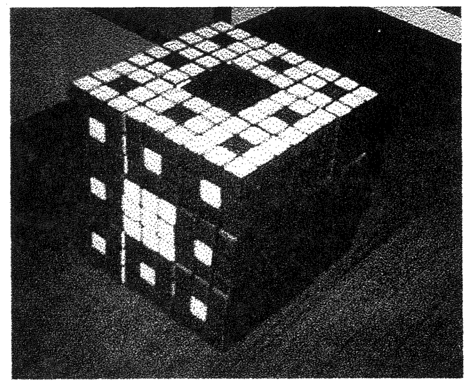

Figure 5. The Menger Sponge Design

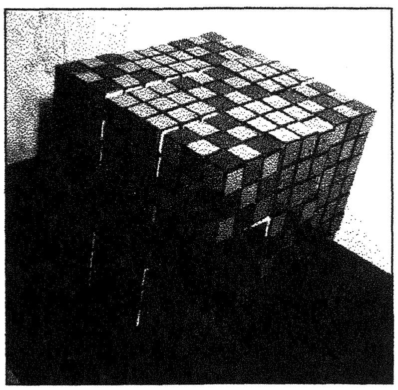

Figure 6a. Box Fractal A

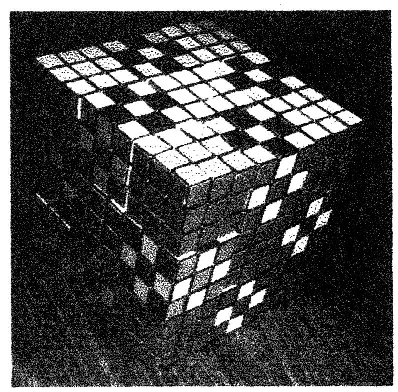

Figure 6b. Box Fractal B

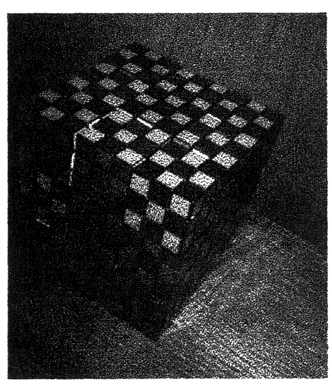

Figure 7: The Checkerboard