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Mathematical Connections
in Art, Music, and Science

On Growth and Form in Nature and Art: The Projective Geometry of Plant Buds and Greek Vases

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Abstract

D'Arcy Thompson's pioneering book *On Growth and Form* showed how a square grid could be laid over the profile of one species of animal and then made to fit that of another related animal by a suitable deformation of space, thus allowing e.g. the shapes of missing bones to be estimated when reconstructing the fossil skeleton of an unknown species. Where Thompson had resorted to quite arbitrary spatial distortions for his examples, George Adams realized that the kind of conformal maps first discussed by Felix Klein (collineations of plane and space with invariant "path curves") would fit at least certain parts of plants and animals by conservative means, being in effect linear transformations in a non-Euclidean setting. This was then extended by Lawrence Edwards to quadratic models, showing how certain pairs of parts of a given plant or animal can be formally related in a species-true manner. I have applied these approaches to Greek amphoræ, showing how their body and base and/or neck shapes are similarly related.

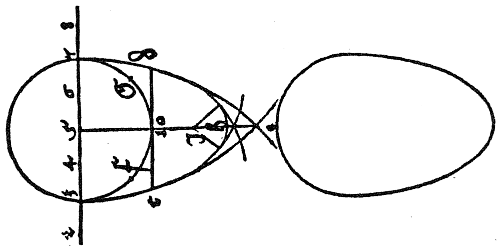
Varying the Canon: Dürer's Manuals Applied by Thompson

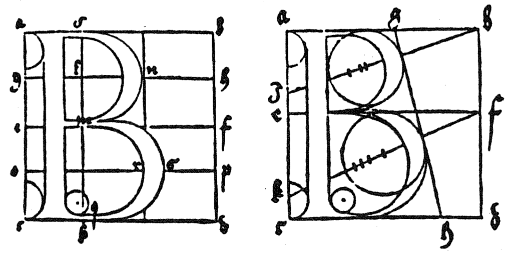
In last year's contribution to these proceedings [1, p. 130], I showed how ancient Egyptian artists used a square grid laid over a sketch both to determine the ideal proportions of the various figures in the scene and to facilitate enlargement of the sketch to full size on a similarly gridded wall. Their name for this procedure, 𓐍𓂋𓏏 = kh·r·t (possibly vocalized *kharet*), has come down to us via Greek *khartē* and Latin *charta* for the "card" of papyrus on which such a "chart" was drawn, augmented in Italian to *cartone* to become the "cartoons" drawn by Renaissance artists such as Leonardo DaVinci and Michelangelo Buonarroti as preliminary sketches for large paintings.

That both uses — determination of proportions and enlargement of sketches — were still practiced in 16th century Italy is evinced by the technique manuals published by German artist Albrecht Dürer upon his return from study there in the first decade of that century. Prior to his publications, only guild members had been allowed to be taught the technical methods of such a trade. For example, at a meeting of architects and builders at Regensburg in 1459 it was resolved that "no worker, master, polisher, craftsman; no one, no matter what he is called, unless he belongs to our trade organization, shall be taught how to build or erect structures from a ground plan" [2, p. 7]. When no less than the builder of Regensburg cathedral published a little treatise on the design of ornamental towers in 1486 he was severely criticized by his guild for this breach of professional confidence, but a precedent had been set.

Of the two traditional uses of the grid, the determination of proportions had become much more complicated by Dürer's era, encorporating numerous subdivisions of each part of the body. Nevertheless, inspection e.g. of the ideal man in Book III of his *Four Books of Human Proportions* [3, folio R verso] lets us verify that, if the distance from ground to hairline is taken as 18 units, then 6 units is still *ob dem knye* (on the knee), 9 is still the wrist (lined but not labeled), 12 still *in der weichen* (in the waist), ca. 14.4 still the armpit (unlabeled), and while 16 is labeled *kin* (chin) rather than shoulder (both of these being in relaxed lowered attitude compared with the Egyptians' stiffer uprightness) the same line actually runs across the top of the shoulder at level of the collar bone. All five key points of the Egyptian canon are therefore recognizably still observed.

But when Dürer attempted to describe the general form of a bird's egg in Book I of his *Painter's Manual* [2, p. 78], he employed the same *ad hoc* means he would apply at length in Book III for shaping Roman letters, going back to the ultimate Roman authority on such matters, Vitruvius. That is to say, he pieced the egg form together quite arbitrarily out of arcs of various circles, in the same manner as a calligrapher pieces together the several parts of a capital letter [2, p. 262].





While such results may be æsthetically pleasing to some extent, their technique remains highly arbitrary, useful in its adaptability much the way cubic splines and wire-framing have become in modern computer graphics, but without any power to convince that the description thereby given of the egg is anything more than superficial — there is no appeal to biology, no connection with the egg's conception in the mother bird's ovary and propulsion, growing all the while, along her oviduct muscle to its eventual laying from her uterus, no "natural philosophy," no science.

When Thompson tried to describe "the shapes of eggs and certain other hollow structures" [4, p. xiii], there was scarcely more science involved. The descent along the oviduct was described merely as by means of "peristaltic waves" (formless in themselves), having the supposed result of blunting the foremost end, since eggs were known to be laid blunt-end first. Even as Thompson wrote this in 1917, there was mounting evidence that later proved conclusively in 1951 that eggs in fact travel pointed-end first along the oviduct and are only turned around afterwards in the uterus before laying [4, p. xiv]. The section on egg forms was accordingly one of those deemed better deleted from the 1961 abridged edition as no longer scientifically supportable, thereby begging the question as to the true, natural, form of the egg, and the role of the oviduct in shaping it.

Studying Invariance: Klein's Lectures Realized by Adams

As it happened, the necessary mathematics to address this question was already at hand, but hidden in the part of Felix Klein's lecture legacy which the fickle fates of two world wars conspired to prevent from being translated. His three volumes on elementary mathematics for training of *Gymnasium* (high school) teachers have long been available in English, but his two further volumes on non-Euclidean geometry and higher geometry for *Hochschul* (university) research have remained relatively inaccessible in German.

Two German-speaking mathematicians (one Swiss, the other Anglo-German) were aware of these latter lectures by Klein. One had even had the advantage of asking Rudolf Steiner (a graduate of Vienna's technical university and editor of natural scientific portion of Kürschner's complete edition of Goethe's works, among many other things) for his advice in attempting to model the egg form and been told (ca. 1924, in a conversation) that he would need to study Lobachevsky space. But this was the pure mathematician, Louis Locher-Ernst, whose career kept him busy with professorial duties at the universities of Winterthur and Zürich, and he did not pursue either the natural science questions or the connection with Lobachevsky space. He did write a study of projective geometry [5] including Klein's approach to classifying distance measurement according to whether the cross-ratio involved was with respect to a pair of fixed points which were real and distinct (multiplicative Lobachevsky metric), real and coincident (additive Euclidean metric), or complex conjugate (angular Riemannian metric). This was the heart of Klein's Erlangen Program: *studying transformations by determining the elements which remain invariant under them*. But when he later attempted a study of egg forms in profile [6, pp. 99-105], he restricted himself to eggs which were even more *ad hoc* constructs than Dürer's — arbitrary free-hand ovals.

The other man, George Adams, had done interpreting for Steiner in England but the subject of eggs apparently never came up. Adams' academic background was in physical chemistry (Cambridge), but left him unsatisfied. Like Locher-Ernst, he had grown up loving to hike in the mountains; unlike Locher-Ernst, he wanted to pursue his mathematics in them, not get away from it. Unaware of Locher-Ernst's work, he wrote his own version of a didactic study of projective geometry, using many illustrations from art history to show how human perception of space had gradually changed over historical time. In it [7, pp. 198 ff. & 434], he mentions Klein's *W-Kurven* but does not yet pursue them further. These were intended by Klein to be *Wurf-Kurven*, referring to von Staudt's theory of harmonic "throws," but Adams creatively re-interpreted them as *Weg-Kurven* from their occurrence as *continuous limit orbits of iterated projective transformations*, differentially applied in ever smaller steps as studied by Sophus Lie, and they have remained known as "path curves" in the English literature of those continuing Adams' work. It took a day in Regent's Park, breathing in the spring air of peace after W.W.II, for Adams to suddenly become aware that the buds on the bushes all around him were 3-dimensional cases of just such path-curve forms, living in nature, to which he later added path-curve descriptions of egg shapes as similarly symmetrical, venturing as far as describing musculature of the left ventrical of the heart as asymmetrical variation of the same basic bud or egg form, and taught these things to Lawrence Edwards, who had had his university studies cut short by W.W.II the way Adams' had been curtailed by W.W.I. The distance metric involved in these profile models was multiplicative like that illustrated last year in our study of the Egyptian canon [1, pp. 126-130] — i.e. it was Lobachevskian. Steiner had been right, but it was the man without benefit of this tip who found it, seen as "observation" (the original sense of Greek *theōrēma*) out of doors, in nature.

The Three Projective Scales

The three kinds of scales at which we looked last year were characterized by their terms forming arithmetic, geometric, and harmonic sequences, respectively, as sensed to be equispaced by human touch, hearing, and sight [1, p. 128]. If a,b,c is a subsequence, they satisfy $b = \frac{1}{2}\cdot(a+c)$ additively, $b = (a\cdot c)^{\frac{1}{2}} = \sqrt{ac}$ multiplicatively, and $1/b = \frac{1}{2}\cdot(1/a+1/c)$ inverse-additively, respectively.

This year's trio is related but different, arrived at by study of transformations and what they leave invariant, following Klein's Erlangen Program. At first, we will find the geometric or multiplicative case now at one extreme, while the arithmetic and harmonic become conflated as central watershed case, and an angularly rotational case appears as new other extreme. But from a deeper view we will come to recognize that all three are multiplicative and all three are rotational when their appropriate arithmetics and geometries are recognized, and all three are related to human sight.

The basic phenomenon of projective geometry as study of perspective vision is that moving objects do not retain the same apparent size; approaching they seem to loom larger, and retreating they dwindle. Furthermore, rotating objects do not retain the same apparent proportions of sizes; the short end of a rectangle seems longer when turned toward us, while the longer sides taper in foreshortening. *It is only when one goes to a third level of comparison, measuring proportions of the apparent proportions* (as measured in turn in proportion to units on some standard foot or meter stick laid onto the pictorial image from without) *that invariance is attained.* If M,A,B,N are four points along a line, then $[(B-M)/(A-M)]/[(B-N)/(A-N)]$ is constant, no matter how one views the line in perspective, or projects it onto other lines; the transformed points will have the same ratio of ratios, known as the cross-ratio of A and B with respect to M and N, abbreviated $\{A,B;M,N\}$.

1. The Hyperbolic Case (Lobachevsky)

In the sequence of canonical Egyptian measurements studied last year 0, ..., 6, 9, 12, $14\frac{2}{5}$, 16, ..., 18 (then associated respectively with ground level and heights to knee, waist, elbow, armpit, shoulder, and hairline), we may take ground and hairline as M = 0 and N = 18, and investigate the cross-ratios of successive pairs of other measurements with respect to them. Taking A = 6 and B = 9 yields $\{6,9;0,18\} = [(9-0)/(6-0)]/[(9-18)/(6-18)] = (9/6)/(-9/-12) = (3/2)/(3/4) = (1/2)/(1/4) = 2$. Similarly, taking A = 9 and B = 12 yields $\{9,12;0,18\} = (12/9)/(-6/-9) = (4/3)/(2/3) = 4/2 = 2$, and the same for $\{12,14\frac{2}{5};0,18\}$ and $\{14\frac{2}{5},16;0,18\}$; they all yield 2 as cross-ratio. When parallel projected as shown in [1, p. 127] so that M remains finitely placed but N is sent infinitely far away (in Euclidean view), then the intermediate points are found to be spaced as powers of 2, measured from new M as 0, with arbitrary choice of which step is the unit as 0^{th} power.

Algebraically, there is a linear fractional transformation or l.f.t. $f(x) = (ax+b)/(cx+d)$ which sends successive scale points to one another. In the case of the Egyptian sequence $6, 9, 12, 14\frac{2}{5}$, 16 it is $f(x) = 36x/(x+18)$. Since such a ratio is unique up to proportion of coefficients, it is clear that any one of the four coefficients a,b,c,d (provided non-0) may be taken as unit and three equations in three unknowns solved to determine the other three. *Any three points of a line may thus be sent to any three other points of that line*, of which this is the special case with $6 \to 9 \to 12 \to 14\frac{2}{5}$ as our choice of scale steps; once these three moves have been made, our three degrees of freedom are used up but it can be verified that $14\frac{2}{5} \to 16$ also works, i.e. that $f(14\frac{2}{5}) = 16$. Setting $f(x) = x$ evidently results in a quadratic equation in x, so there will necessarily be *two fixed points* or *invariants*, namely the roots of this 2nd degree equation. In this case, setting $36x/(x+18) = x$ yields $36x = x^2 + 18x$, $x^2 - 18x = x(x-18) = 0$, whence $x = 0$ and 18 are the *real and distinct* invariants. Applying the transformation repeatedly yields $16 \to 16.94 \to 17.45 \to 17.72 \to 17.86 \to \cdots$ (to 2 place accuracy) approaching 18 as limit but never reaching it; values $x > 18$ are also drawn toward 18 by f so that 18 is said to be an "attractor" or a sink of f, while values $\pm x \approx 0$ move away from 0 as "repeller" or source. Applying the inverse transformation $f^{-1}(x) = 18x/(36-x)$ reverses these roles, sending $6 \to 3.6 \to 2 \to 1.06 \to 0.54 \to \cdots$ closer to 0 as sink and away from 18 as source.

Analytically, as we saw last year [1, p. 130], if we follow this f(x) by a further $g(x) = (x-9)/9$, then the sequence $0, \ldots, 6, 9, 12, 14\frac{2}{5}, 16, \ldots, 18$ is sent to $-1, \ldots, -\frac{1}{3}, 0, \frac{1}{3}, \frac{3}{5}, \frac{7}{9}, \ldots, 1$ which are the values of $\tanh ku$ for $k = -\infty, \ldots, -1, 0, 1, 2, 3, \ldots, \infty$ and $u = \frac{1}{2}\ln 2$. The sequence is thus not only projectively equivalent to powers of 2, terms of a *geometric sequence* generated by *repeated multiplication* by 2 as predicted by $\{6,9;0,18\} = 2$; it is also projectively equivalent to *hyperbolic rotation* through angle $u = \frac{1}{2}\ln 2$. This is what led Klein to define the distance from A to B, with respect to M and N as real and distinct inaccessible points, as $\frac{1}{2}\ln\{A,B;M,N\}$; it is the distance metric belonging to 1-dimensional Lobachevsky space [8, p. 164].

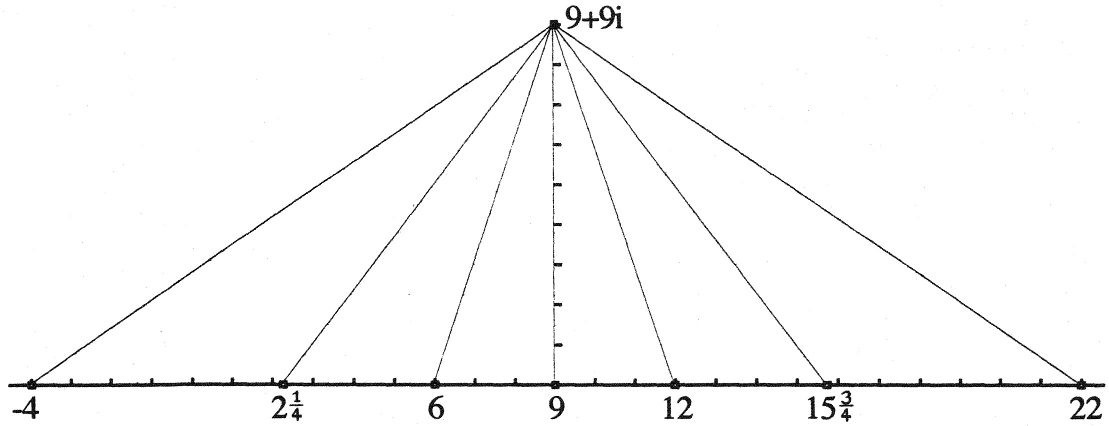
2. The Parabolic Case (Euclid)

Suppose instead that the scale consisted of $\ldots, 6, 9, 12, 15, \ldots,$ terms of an *arithmetic sequence* generated by *repeated addition* of 3, so its l.f.t. is $f(x) = x+3$. Setting $f(x) = x$ yields $x = \pm\infty$, identified projectively as *real and coincident* point $M = N$, so the cross-ratio of such a sequence (and any to which it is projectively equivalent, including harmonic sequences obtained by inversion) must be 1, since numerator and denominator of $[(B-M)/(A-M)]/[(B-M)/(A-M)]$ are now identical. But $\frac{1}{2}\ln 1 = 0$, so the corresponding notion of *parabolic rotation* must be through a vanishingly small angle about that infinitely remote point, successive "radii" appearing as parallel lines. This is the distance metric belonging to 1-dimensional Euclidean space, whose unit size is arbitrary (thus the necessarily free choice of conventions for inches, centimeters, etc.). There are also almost-but-not-quite trivial parabolic versions of trigonometric functions given by $\mathrm{sinp}\, a = \mathrm{tanp}\, a = a$ and $\mathrm{cosp}\, a = 1$ which satisfy the analog of DeMoivre's theorem etc., argument $2a = y$ = double area of right triangle with corners (0,0), (1,0), (1,y), analogous to viewing 2u as double area of hyperbolic segment with corners (0,0), (x,0), (x,y), with $x^2 = 1$ playing the role of unit parabola analogous to $x^2 \pm y^2 = 1$ as unit circle or hyperbola. There is even a short-and-sweet (2-term) Fourier expansion analog available by taking $e^{2\phi a} = \mathrm{cosp}2a + \phi\,\mathrm{sinp}2a = 1 + 2a\phi$ if ϕ is a nilpotent $\phi^2 = 0$ causing all higher power terms to vanish, analogous to taking $e^{2\varepsilon u} = \cosh 2u + \varepsilon \sinh 2u = 1 + 2u\varepsilon + \frac{1}{2}(2u)^2 + \frac{1}{6}(2u)^3\varepsilon + \cdots$ with $\varepsilon^2 = 1$. If ε is the real unit 1, then we have Klein's $\frac{1}{2}\ln e^{2u} = u$; if ε is something else (one of the Pauli matrices, like i as one of the quaternions) then we must modify the definition to be $u = (\frac{1}{2}\ln e^{2\varepsilon u})/\varepsilon$, and similarly in the parabolic case $a = (\frac{1}{2}\ln e^{2\phi a})/\phi$, both just like the formula $\theta = (\frac{1}{2}\ln e^{2i\theta})/i$ for circular rotation first discovered by Laguerre [9, p. 158].

3. The Circular Case (Riemann)

Suppose lastly that the scale consisted of $\ldots, 6, 9, 12, 15\frac{3}{4}, \ldots$. The l.f.t. granting these three movement wishes is $f(x) = (18x+162)/(-x+36)$, or $18(x+9)/(-x+36)$. Setting this $f(x) = x$

yields a quadratic equation $x^2 - 18x + 162 = 0$ with *complex conjugate* roots $x = 9 \pm 9i$. Plotting them in the complex plane, we recognize them as the vantage points from which the terms of the sequence subtend *equal angles* of *circular rotation.* What is the angle? Evaluating the cross-ratio {6,9; 9+9i,9–9i} is laborious, but careful work shows it to be $\frac{4}{5} + \frac{3}{5}i = \cos 2\theta + i \sin 2\theta = e^{2i\theta}$ for $2\theta = \tan^{-1}\frac{3}{4}$, whence $\theta = \tan^{-1}\frac{1}{3} \approx 18.4349\ldots°$, as seen in the diagram below showing the fixed point 9+9i at center of counterclockwise motion (which would be mirrored by clockwise motion about other fixed point 9–9i), confirming Klein's modified definition that $\theta = (\frac{1}{2}\ln\{A,B;M,N\})/i$.



It is even possible to find the analog of the earlier construction which revealed powers of 2: As before, the idea is to send one fixed point to a new 0, the other to ∞, and one of the intermediate scale points (arbitrary which) to the new 1 (as 0^{th} power). Instead of using parallels, however, we use the trick familiar to students of complex analysis whereby any three points are easily sent to 0,1,∞ by a conformal map (another name for l.f.t.). Here, we send x = 9+9i, 9, 9-9i to 0,1,∞ via $g(x) = (x-9-9i)/(9-9i-x)$. The sequence ..., 6, 9, 12, 15¾, 22, ... is then found to be mapped to $(\frac{4}{5}+\frac{3}{5}i)^k$ for k = ..., -1, 0, 1, 2, 3, ... as result of *repeated multiplication* by the base predicted by the cross–ratio {A,B;M,N}, appearing now as rotation through $2\theta = \tan^{-1}\frac{3}{4}$ around unit circle about 0.

It had been the suggestion of Christian von Staudt (in his pictureless 1847 *Geometrie der Lage*) that *one could in effect picture otherwise invisible imaginary or complex elements by taking them to be the implied invariants of a circling motion among the visible real elements,* thus anticipating (and perhaps even suggesting) Klein's Erlangen approach to studying transformations of all kinds. The general form of a linear fractional transformation that leaves complex conjugate elements M,N = m±ni invariant and moves any real element A = x to B = f(x) in such a way as to subtend angles $\theta = \pm 180°/n$ from M,N when viewed in complex plane is given by the ratio

$$\frac{(-m + n\cot\theta)x + m^2 + n^2}{-x + m + n\cot\theta}.$$

Since any n-cycle would serve the same purpose of locating a given set of complex conjugate invariants, in practice it is common to use the n = 2-cycles or involutions satisfying $f^2 = id$. For since $\theta = 90°$ in that case and $\cot 90° = 0$, these have the simpler form $f(x) = (m^2 + n^2 - mx)/(m-x)$.

Taking $u = \frac{1}{2}\ln k$, $m = \frac{1}{2}(M+N)$, and $n = \frac{1}{2}(N-M)$, it is also possible to give an analogous formula for the general l.f.t. that leaves real and distinct elements M,N = m∓n invariant and moves any other real element A = x to B = f(x) so as to yield {A,B;M,N} = k as base of power sequence:

$$\frac{(m + n\coth u)x - m^2 + n^2}{x - m + n\coth u}.$$

[These details, while fiddly, are given here as Bridge between the usual elementary and advanced expositions which leave them out (as too advanced or too elementary). Anyone who wishes may skip them, but it is hoped that anyone attempting to work in the field will find them useful.]

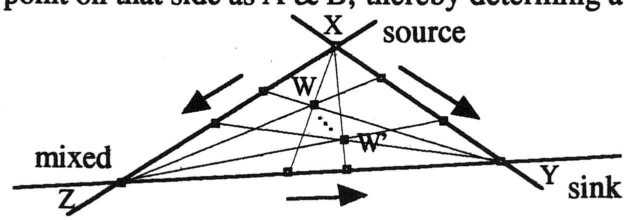
Path Curves as Invariants of Collineations in 2 and 3 Dimensions

Non-guild-members of 15th century Regensburg were not allowed to "be taught how to build or erect structures from a ground plan," but we are about to learn how to do just that! The ground plans and elevations of the buds and eggs we seek to erect will be formed in the simplest possible way. First, we will draw the equivalent of a square grid on each of two charts, then declare each bud or egg's outlines to be the straight lines traversing those grids diagonally, like bishops' moves on a chessboard. The geometry will appear to be curvilinear, to make the bud or egg properly oval; but the algebra will remain linear, restricted to 1st degree occurrences of each variable. The catch — what makes this magic possible — is the word "equivalent"; for instead of the usual Cartesian grid with x and y axes that bear Euclidean or arithmetically-measured scales, we shall be using homogeneous x,y,z coordinates for the projective plane and their axes will bear one or other of the two non-Euclidean scales: Lobachevskian or geometrically-measured for the elevation and Riemannian or angularly-measured for the plan. The bud with its petal windings and the egg with its oviduct musculature will then arise in 3-D by their interweaving.

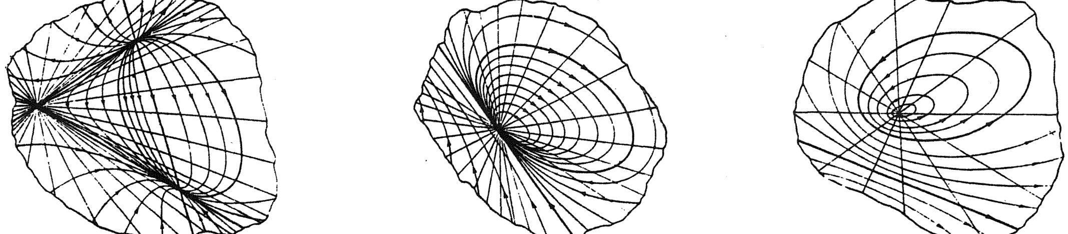
Instead of 3, we now have 4 degrees of freedom: *Any* 4 *points of the plane may be moved to any other* 4 *points*, and a unique collineation of the plane can be found which carries this out, moving every other point one-to-one to another point, every straight line to another straight line.

Instead of 2, there will now be 3 invariant points, joined pairwise by 3 invariant lines, forming an *invariant triangle*. Each of its corners is fixed in place by the collineation, while lines through them (except for the two side lines apiece) will move; likewise each of its sides is fixed in place, while points on them (except for the two corner points apiece) will move.

Rather than moving 4 points arbitrarily and then hunting for the 3 fixed points, it is easier to use 3 our our 4 degrees of freedom by specifying the fixed points X,Y,Z to begin with, and then moving one further point (not collinear with any two of X,Y,Z) from W to W'. This fourth point casts a moving shadow on each side of the invariant triangle, when projected from the opposite corner. The two corners on each side can be taken in turn as M & N, and projections of the two positions of the moved point on that side as A & B, thereby determing a unique l.f.t. on each side.

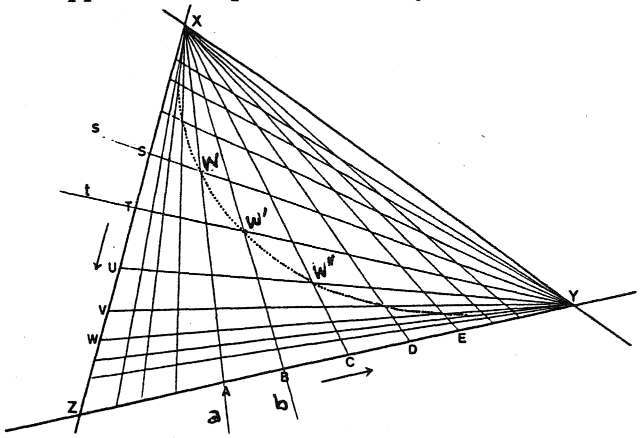


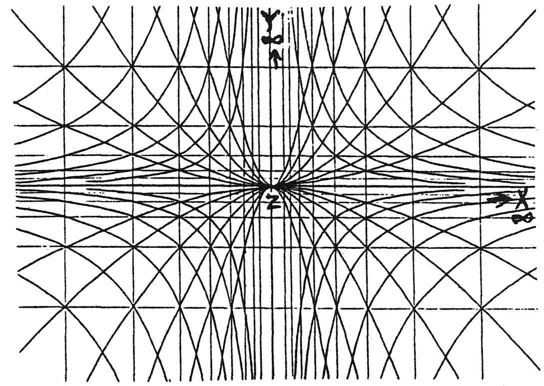
The three parts of the invariant triangle, however, are not all alike! The induced motions along two sides (here XY and XZ) flow away from their common corner (X), making it a source; those along two other sides (XY and ZY) flow toward their common corner (Y), making it a sink; while those on the remaining pair of sides (XZ and ZY) flow contrarily, making their common corner (Z) of mixed nature. The induced scales along the two sides through the mixed corner are used to create a fan of lines apiece from the source and sink corners. They form the non-Euclidean grid or chessboard, and the further (and prior) positions of W,W',W",... are traced along along it diagonally (ultimately with differential refinement to become smooth), forming stages of a *path curve* that departs tangentially from the source and moves tangentially toward the sink. Depending on nature of source and sink, the resulting trajectories look like one of the following [8, p. 105]:



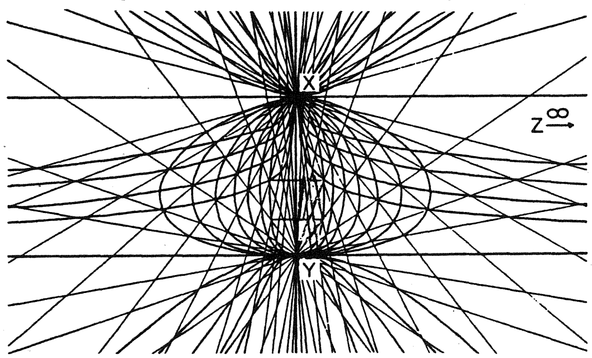
X & Y real & distinct   X & Y real & coincident   X & Y complex conjugate

Klein's illustrations above [loc. cit., repeated on p. 205], however, show one fan of lines from mixed corner Z (which remains real in every case). The interweaving of the two fans from source X and sink Y is shown by Edwards [10, p. 206, or 12, p. 37], below left. He also shows what happens if Z is placed centrally and X & Y are sent to ∞ orthogonally, below right [12, p. 40].

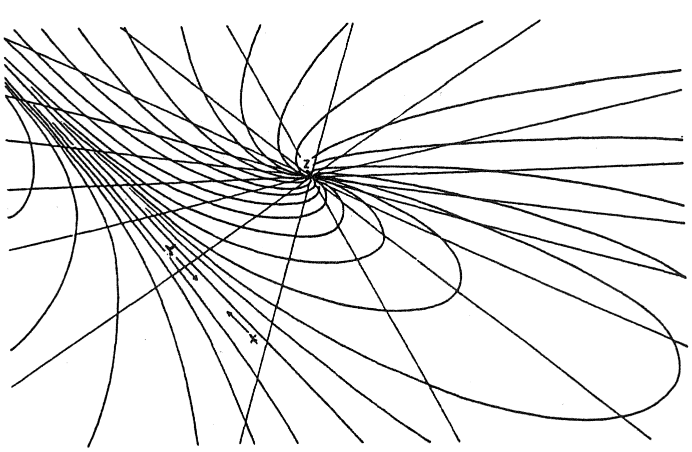


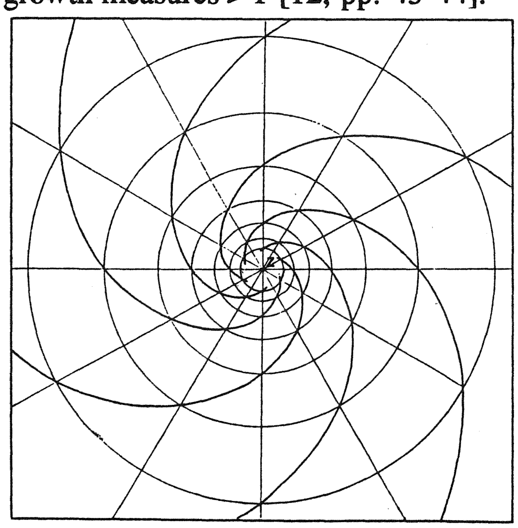


As special cases, the curves can be ellipses when Z is remote and ZX//ZY, or hyperbolæ when Z is central and ZX⊥ZY; in both cases, to be conic sections, the growth measure (the constant k at base of repeated multiplication) must be the same along both scales through Z. *In general, when different growth measures occur along these two sides, the two ends of the ovals will be different, one more or less pointed and the other more or less blunted* — both more or both less so, not independently choosable. Unlike Dürer's method, the entire oval is determined in one piece. It is also not unique but a member of a family of similar forms, affinely stretched or compressed [12, p. 42].



For the plan view, we must also generalize Klein's special case. Instead of ellipses or circles (when X,Y become I,J, the "circling points at ∞") with k = 1 preserving projective radii, general spirals that become logarithmic spirals arise from radial growth measures > 1 [12, pp. 43-44].





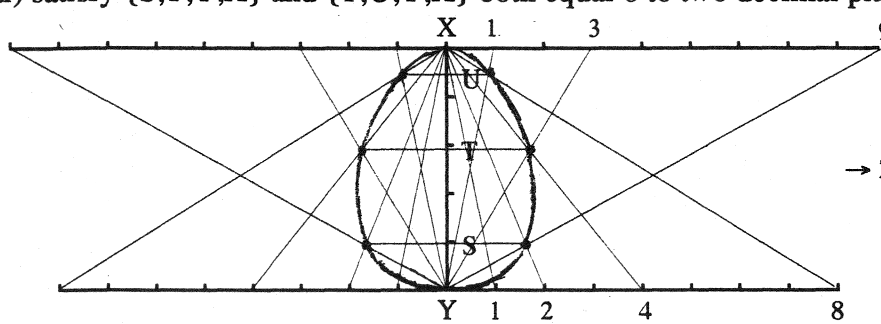
To make good on the claim that all of these path curves may be considered "straight" lines only made to look curved by their non-Euclidean settings, we must verify that they are all given by algebraically linear equations in their coordinates. These must be homogeneous coordinates, i.e. triples (x,y,z) or (x_1,x_2,x_3) only determined up to proportionality, with (x,y,1) corresponding to Cartesian (x,y) for finitely-placed points and (x,y,0) corresponding to points on the inaccessible line at infinity.

Conic sections with Cartesian equations such as $xy = C$ for orthogonal hyperbolæ, $y = Cx^2$ for parabolæ, and $x^2+y^2 = C$ for circles seem quite different, but Klein [9, pp. 168-169] points out that all three are of homogeneous form $x^a y^b z^c = C$ with $a+b+c = 0$, hence all have linear equations $a\ln x + b\ln y + c\ln z = 0$ in the logs of those coordinates. The trick is to view all three as of form $xy = Cz^2$, pictured as above drawn tangent to corners X and Y of the invariant triangle, avoiding Z. For the hyperbolæ, it is the homogeneous z variable which is fixed = 1 to become Cartesian; for the parabolæ, it is either x or y that is so frozen; for the circles, we first factor x^2+y^2 as $(x+iy)(x-iy) = r^2$, then rename these as $x_1 x_2 = C(x_3)^2$, satisfying $(x_1)^1(x_2)^1(x_3)^{-2} = C$ with $1+1-2 = 0$.

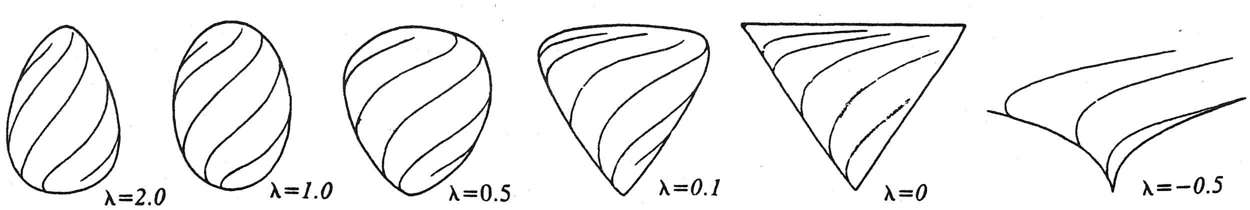
We can also factor the Cartesian logarithmic spiral equation $r = (x^2+y^2)^{\frac{1}{2}} = (x+iy)^{\frac{1}{2}}(x-iy)^{\frac{1}{2}} = Ce^{k\theta}$. But $x+iy = re^{i\theta}$ and $x-iy = re^{-i\theta}$, so $(x+iy)/(x-iy) = e^{2i\theta}$ and $[(x+iy)/(x-iy)]^{-\frac{1}{2}ki} = e^{k\theta}$. Substitution into the previous equation gives $(x+iy)^{\frac{1}{2}}(x-iy)^{\frac{1}{2}} = C[(x+iy)/(x-iy)]^{-\frac{1}{2}k}$, whence homogeneously $(x+iy)^{\frac{1}{2}(1+ki)}(x-iy)^{\frac{1}{2}(1-ki)}t^{-1} = C$, of same form $(x_1)^a(x_2)^b(x_3)^c = C$ with $a+b+c = 0$.

Bud, Egg, and Vortex Shapes

The equation $a+b+c = 0$ above is equivalent to $e^a e^b e^c = 1$ which we may rename $\alpha\beta\gamma = 1$ with $a = \ln\alpha$, $b = \ln\beta$, $c = \ln\gamma$. In the all-real case for ovals, this means [cf. 12, pp. 271-272, where they are inconsistently labeled] that if α is the multiplier or growth constant along XZ and β is that long ZY, then the multiplier along YX must be $\gamma = 1/\alpha\beta$ to satisfy $\alpha\beta\gamma = 1$. This is true if X,Y,Z take turns being the source along each side in cyclic fashion, as stated. When re-stated so that X and Y are respective sources along XZ and YZ but Y is source along YX, then it is $\gamma = \alpha\beta$. For example, suppose $\alpha = 3$ and $\beta = 2$, so that we may take successive powers of those bases as scale steps along XZ and YZ when these axes are parallel and YX is perpendicular to them. Then if the distance from Y to X is taken e.g. as 5 of the same units we can confirm that points S = 0.94, T = 2.90, U = 4.46 along YX (corresponding to heights of three successive steps along a typical path curve oval) satisfy {S,T;Y,X} and {T,U;Y,X} both equal 6 to two decimal places.



To express the relative pointed- and bluntedness of such an oval, Adams and Edwards use parameter $\lambda = a/b = \ln\alpha/\ln\beta$ [loc. cit.]; here, $\lambda = \ln3/\ln2 = 1.585$. If $\lambda = 1$, both ends are alike as ellipse; $\lambda \to \infty$ or 0 makes ovals tend to triangles, while $0 > \lambda > -1$ makes them vortical [12, p. 53].



The other parameter used to describe the steepness of pitch of the spiral winding about the o-void form in 3-D *is* $0 \leq \varepsilon \leq \infty$, yielding the transition from the analogs of horizontal latitude circles, to loxodromes spiralling from one pole to other, to vertical longitude ovals between poles, defining 2ε to be the growth from Y to X during one radian of spiral winding. To compute it [cf. 12, pp. 316-317, where cross-ratio at bottom of p. 316 is upside-down], we first find the multiplier μ of upward growth between any two points along the vertical axis (Edwards uses two points lying at about the middle 3/4 of the bud height) by evaluating a cross-ratio such as {T,U;Y,X} in the illustration above, then take $m = \ln\mu$. Next, we measure the distances of the two spiral points from their projections onto the vertical axis, setting them each in ratio to the radius at its level and adding their inverse sines to determine the total angle turned from one to the other, expressed in radians as ω; then $\varepsilon = \frac{1}{2}m/\omega$. With these two parameters, we have complete control over the curves.

A wide variety of plant forms such as flower buds, inflorescences, leaf buds, and seed cones were intuited by Adams to be path curve forms. After Adams' death in 1963, Edwards set about testing this hypothesis, both in his Scottish home and abroad in Australia; others have done some field work in England, Canada, and the U.S., but it is Edwards (now equipped with camera and computer link) who continues to make daily rounds, inspecting e.g. leaf buds on the same dozen or so trees from their formation in autumn to opening in spring, year after year, documenting not only their forms throughout that time but also the slight but statistically significant regular pulses in the λ values expressing those forms (a fortnightly "Is it spring yet?" incipient opening gesture, tied in to shifting lunar rhythms — cf. [12, Ch. 15], supplemented by several volumes of subsequent data analyses) akin to animal heart-beats (taking left ventrical as asymmetrical path form — [cf. 12, Ch. 8]) that lead too far to attempt to describe further here.

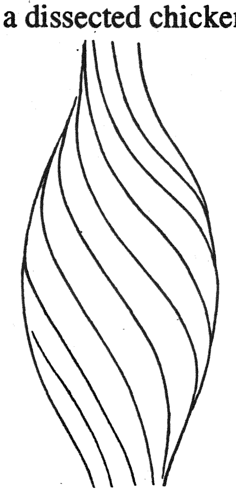
Suffice it to exhibit the seed cones of a Scots pine (with $\lambda = 3.03$, $\varepsilon = 0.22$, and Mean Radius Deviation = 1.3%) at left and a larch ($\lambda = 1.7$, $\varepsilon = 0.23$, MRD = 0.9%) at right [12, pp. 66-67].

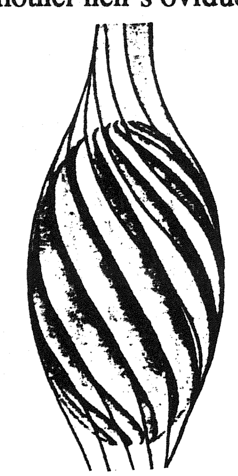
Scots Pine

Larch

One morning, Edwards observed a slight trace of a spiral imprint on the membrane between the shell and hardened albumen of his boiled egg. Visiting the Poultry Institute in Glasgow, he was shown its source in a dissected chicken: The mother hen's oviduct muscle is spirally wound.

A tracing from photograph of partially-formed egg, part-way down oviduct.





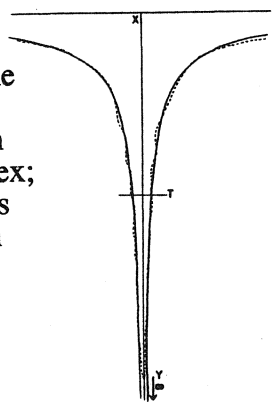
A comparison with the nearest path curve which could be found [12, pp. 178-179].

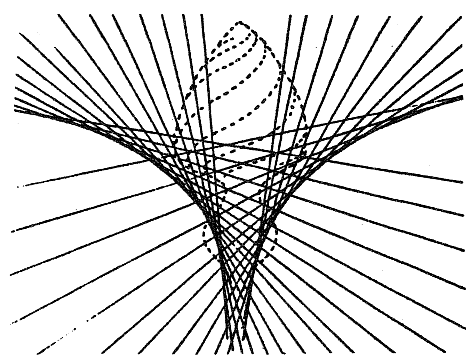
The pitch is less at first (smaller ε), getting gradually more (larger ε) as the egg descends, with the effect that *the widening egg distends the oviduct muscle wall in such a way that the resulting pitch remains virtually constant throughout the descent*. The egg is spirally *wrung* from ovary to uterus (not just peristaltically pushed), the mother's musculature completing the flower bud analogy.

The geometry of 3-D path surfaces is a natural extension of what we have seen so far: There are 5 degrees of freedom, of which we may use 4 to specify the corners of an invariant tetrahedron, allowing a moving further point to project upon opposite faces as invariant triangles of induced 2-D motions, in turn projected upon edges as 1-D motions, determing the basic scales. Ignoring coincident corners, there are three principal cases: all-real tetrahedron, semi-real semi-complex, and all complex (with corners disappearing pairwise as conjugates); in all three cases, the curves pass through two corners and avoid the other two. The first case has no known natural application; the second yields the convex ovoid forms of buds and eggs (with two real tangent planes meeting at horizon and two complex conjugate planes meeting in central axis) and concave vortices; the third yields generalized helical forms, such as the oviduct muscle.

One of Edwards' most profound original discoveries is the *pivot transformation* whereby the planes tangent to a family of vortex spirals coaxal with a family of ovoid spirals are allowed to pick out the points of the latter family to which they are tangent, yielding a family of path surfaces of second degree, one member of which for a rose bud yields a rose hip, and one member of which for a St. John's wort bud yields a St. John's wort gynæcium, species-true, that also lead too far to describe further here [cf. 12, Ch's 9 & 10]. Suffice it to note that the kind of vortices required for this purpose are those with $\lambda < -1$ with ideal tips at infinite distance, which Adams intuited matched water vortex shapes and Edwards has experimentally verified by photographing controlled tank flows.

The dotted outline is traced from a photograph of an actual water vortex; the smooth line is path curve match with $\lambda = -1.74$ [12, p. 166].





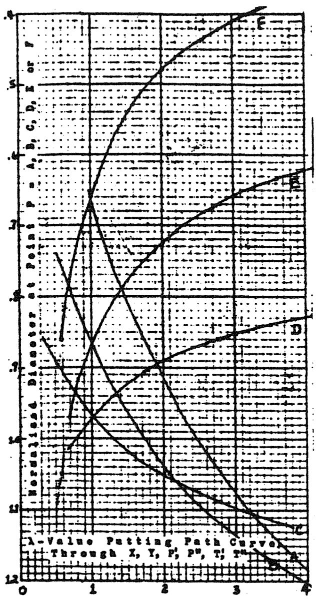
The rose bud as member of ovoid family of curves, and intersection lines of planes tangent to member of vortex family, resulting in rose hip as pivot transform [12, p. 150].

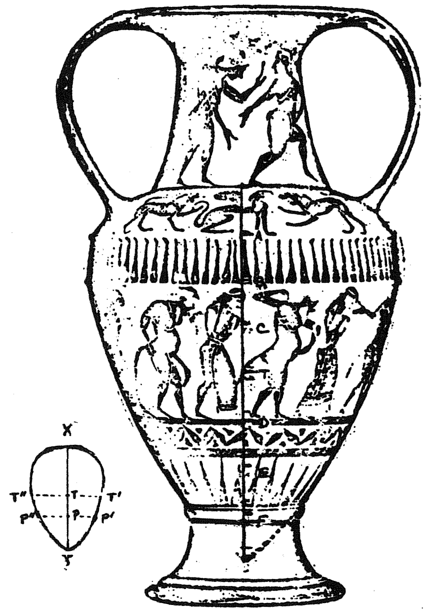
Greek Amphoræ

It was one of Edwards' Australian collaborators, John Blackwood, who suggested at Easter of 1979 that one look into Grecian urns as candidates for path surface shapes. This seemed plausible to me on at least two counts — the Greeks were known to have been sensitive to mathematical æsthetics in their art and architecture, and birds have surely arrived at path shapes for their eggs out of design merit through evolution, making such shapes recommendable to be imitated (whether consciously or not) by humans for their own container designs — so I procured a copy of Arias' *A 1000 Years of Greek Vase Painting* (Abrams, N.Y., 1961) and began investigating.

The first difficulty I ran into was one also encountered by Edwards: Exactly where are the ends of the ovoid? On a leaf or flower bud as well as a seed cone, one end is attached to a stem which obscures its exact location, and the other is likely to be in the process of slightly opening; only eggs have well-defined natural ends. The vase ovoids have stands set into their bottoms and pouring necks set into their tops, interrupting the main form, so like Edwards I had to exercise judgment on selection of ideal end points (extending the given profile by dotted lines, next page).

The simplest way to arrive at a λ value for an oval profile is by means of a nomogram such as that reproduced on next page. The height of the oval is divided into 8 equal parts labeled top to bottom as X,A,B,C,T,D,E,F,Y, and each relative diameter at A,B,C,D,E,F is measured against that at T as unit and found on vertical scale of nomogram; the horizontal coordinate is then the matching λ value which would place a path curve oval through ends X and Y as well as through the profile points P', T' and their mirrors P'',T'', when P is one of points A through F [13, p. 3]. The seven ideal radii in this case for points A to F are 3.73, 3.83, 3.62, 3.23, 2.70, 2.03*, and 1.23.

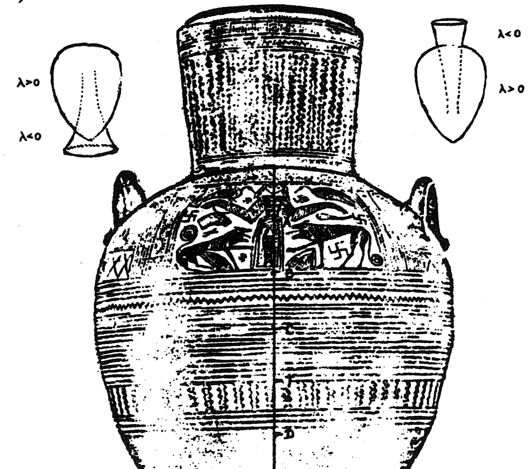




The measured radii are 3.78, 3.875, 3.625, 3.23, 2.71, 2.08*, and 1.23. Dividing by 3.23 at T gives normalized radii 1.170, 1.200, 1.122, 1.0, 0.839, 0.644, and 0.381, leading to λ values 3.86, 3.92, 3.79, —, 3.50, 3.28*, and circa 3.7. Assigning weights 4, 2,1,-,1,2,4 to these values, they average 3.71, with mean deviation .21 or 6%.

The sharp eye can detect (at least after-the-fact) that it is level E (*) which shows the greatest deviation in measure from mean λ value, pointing to a slight excess in width at that level, noticeable particularly on the left side. It is because the two sides of such objects do not always agree (glaringly so in the case of bumpy pine cones but still subtly so for vases despite their being turned smoothly on a potter's wheel) that we first measure diameters and then halve them to find radii.

The next example [13, p. 5] is painted to show a *Potnia Therōn* or Queen of the Wild Beasts. Besides being another excellent path surface shape of ovoid type (whose analysis is shown in fine print: weighted mean λ = 2.02 with mean deviation .08 or 4%), it is noticeable in this case that both stand and neck shapes are also suggestive of path surfaces of vortex types. The previous example can then also be recognized after-the-fact as having at least its base of similar vortical shape though its neck narrowed rather than widened to the top. Scanning through the book's collection, it became evident that many of the vases had the vortical base shapes as standard design feature, while neck designs varied, some narrowing, others widening. A wide-bottomed base is, of course, common sense; *but when inverted we see Edwards' pivot*!



	Ideal	Meas.	Norm.	λ
A	5.31	5.25	.897	1.93
B	6.09	6.10	1.043	2.05
C	6.15	6.20	1.060	2.16
T	5.85	5.85	1.000	–
D	5.20	5.25	.897	1.89
E	4.24	4.25	.7265	2.00
F	2.80	2.75	.470	2.10
			Wtd. Average	2.02
			Mean Dev.	.08
				(= 4%)

Two other examples are shown on the next page, one with a widening neck and the other narrowing, one with a base suggestive of ovoid cap and the other of an arbitrarily tiered design, to give some indication of the variety of neck and base forms extant. The bodies, however, remain very good path surface shapes, that on the left having weighted mean λ = 3.20 with mean deviation .06 or 2%, and that on the right having weighted mean λ = 2.77 with mean deviation .19 or 7%. That on the left is thus a virtually perfect path ovoid. That on the right shows its main deviation from ideal at point A, which may be due to its having handles attached differently at that level and thus confusing my measurements there. The handles are attached below A on the vase at left, above A on the vase at right. But as a little extra fillip, the vase at right has an extant lid with what appears to be a near-perfect path ovoid (inverted, in usual bud orientation) as top-notch.

	Ideal	Meas.	Norm.	λ
A	8.09	8.10	1.109	3.25
B	8.40	8.40	1.151	3.20
C	8.05	8.05	1.103	3.20
T	7.30	7.30	1.000	–
D	6.18	6.24	.855	3.00
E	4.70	4.70	.647	3.20
F	2.95	2.95	.400	3.20
		Weighted Average		3.20
		Mean Deviation		.06 (= 2%)

	Ideal	Meas.	Norm.	λ
A	7.00	7.20	1.067	2.93
B	7.51	7.50	1.111	2.69
C	7.33	7.30	1.081	2.60
T	6.75	6.75	1.000	–
D	5.82	5.875	.870	2.50
E	4.56	4.60	.681	2.61
F	2.89	2.90	.430	2.64

Mean Deviation .19 (=7%) Weighted Average 2.77

If d_1 is the diameter at height h_1 and d_2 that at h_2, measured up from Y = 0 to X = 1, then consideration of similar triangles [cf. 12, p. 307] yields $\lambda = \ln[d_2h_1/d_1h_2]/\ln[d_1(1-h_2)/d_2(1-h_1)]$. If height 1 is that of T and height 2 that of levels A through F in turn (whose diameters we abbreviate by same letters), then weighting by 4,2,1,-,1,2,4 gives $\lambda = \ln 4A/\ln(7/4A)$, $\ln 2B/\ln(3/2B)$, $\ln(4C/3)/\ln(5/4C)$ at heights A,B,C, and reciprocally for D,E,F, ending with $\ln(7/4F)/\ln 4F$ as equations for the six nomogram curves [loc. cit., pp. 308-309].

It was my privilege to be editor of the quarterly *Math.-Phys. Correspondence* from 1973-'83 in which Edwards' geometry and biology papers first appeared, since collected and expanded in book form [10 and 12 — 11 is out of print]. May the present summary and background help to bring that work to the attention of the wider readership it so richly deserves.

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