Bridges
Mathematical Connections
in Art, Music, and Science

A Fresh Look at Number

Jay Kappraff
Department of Mathematics
New Jersey Institute of Technology
Newark, NJ 07102
Kappraff@aol.com

Gary W. Adamson
P.O. Box 124571
San Diego, CA 92112 - 4571

Abstract

A hierarchy of rational numbers is derived from the integers and shown to be related to naturally occurring resonances. The integers are also related to the Towers of Hanoi puzzle. Gray code is introduced as a tool to aid in understanding Towers of Hanoi and also used to predict the symbolic dynamics of the logistic equation of dynamical systems theory. The Towers of Hanoi and Gray code are both generalized to number systems base n and used to derive a probability density function for the divisibility of integers. The number system based 4 expressed in generalized Gray code is shown to be a natural framework for the representation of the 64 codons of DNA.

1. Introduction

What do the divisibility properties of the positive integers have to do with the Towers of Hanoi puzzle, Gray code, dynamical systems theory, and the structure of DNA? This paper explores these relationships.

2. A Natural Hierarchy of Numbers

It is not commonly known that when one counts the positive integers one also counts a hierarchy of rational numbers in lowest terms in which the "most important" rational numbers appear higher on the list. Our meaning of "most important" will be defined below, but first we describe a procedure for determining location of a number in the hierarchy by giving an example. What is the 19^{th} rational number in this hierarchy? To answer this, first write 19 in binary, i.e., $19 = (10011)_2$. Next duplicate the last digit and separate the contiguous 1's and 0's as follows:

1 00 111 corresponds to [1,2,3]

where the numbers in brackets are the number of 1's and 0's in each contiguous group, i.e., 1 one, followed by 2 zeros, followed by 3 ones.

The numbers in brackets are the indices of the continued fraction expansion [1],[2] of 19^{th} rational number in the hierarchy, i.e.,

$$19 \text{ corresponds to } [1,2,3] = \cfrac{1}{1+\cfrac{1}{2+\cfrac{1}{3}}} = \frac{1}{1}\,\frac{1}{+2}\,\frac{1}{+3} = 7/10$$

Note that the boldface indices appear as elements of the continued fraction. This leads to the following algorithm for determining the rational number corresponding to any integer of the hierarchy.

Algorithm 1:

a) Write the number in binary.
b) Duplicate the last digit and write the numbers of 0's and 1's in each contiguous group, referred to as the indices, beginning from left to right.
c) These are the indices of the continued fraction expansion of the rational number in lowest terms, i.e., $p/q = [a_1, a_2, \ldots, a_n] = \frac{1}{a_1} + \frac{1}{a_2} + \ldots + \frac{1}{a_n}$

Note that $[a_1, a_2, \ldots, a_n] = [a_1, a_2, \ldots, a_n - 1, 1]$. By duplicating the last digit of the binary notation we have chosen the first rather than the second representation. This procedure can also be carried out in reverse to determine the hierarchy number of a given rational number. Using this algorithm 1/2 = [2] corresponds to the integer 1 after eliminating the duplicated last digit. Therefore 1/2 sits atop the hierarchy. Table 1 lists the first 15 numbers in the hierarchy. In this Table, the 2^{n-1} integers with n digits are grouped in blocks and their corresponding rational numbers have continued fraction representations with indices that sum to n+1 ,e.g, there are 8 integers with 4 digits whose corresponding rationals have indices that sum to 5.

Table 1. A Hierarchy of Rational Numbers and their Representations as Binary, Gray Code, and Tower of Hanoi Positions

N	Binary	Gray	Moduli		Indices	Fraction	Pegs A	B	C	TOH
		3210								
0	0	0			[0]	0/1	(Start)			
1	1	1	0		[2]	1/2	1			1
2	10	11	1		[1,2]	2/3	1	2		2
3	11	10	0	3	[3]	1/3		1/2		1
4	100	110	2		[1,3]	3/4	3	1/2		3
5	101	111	0	3	[1,1,2]	3/5	1	3	2	1
6	110	101	1	5	[2,2]	2/5	1	2/3		2
7	111	100	0	3	[4]	1/4	1/2/3			1
8	1000	1100	3		[1,4]	4/5	1/2/3	4		4
9	1001	1101	0	3	[1,2,2]	5/7	2/3	1/4		1
10	1010	1111	1	5	[1,1,1,2]	5/8	2	3	1/4	2
11	1011	1110	0	3	[1,1,3]	4/7	1/2	3	4	1
12	1100	1010	2	7	[2,3]	3/7	1/2	3/4		3
13	1101	1011	0	3	[2,1,2]	3/8	2	1	3/4	1
14	1110	1001	1	5	[3,2]	2/7	1	2/3/4		2
15	1111	1000	0	3	[5]	1/5		1/2/3/4		1

It should be noted that, strictly speaking, it is the blocks in Table 1 that are ordered in the hierarchy. Within each block there is no strict ordering of "importance," e.g., in block 4, 3/4 is no more "important" than 1/4, 3/5, or 2/5. The numbering of rational numbers within each block of Table 1 follows the well-known Farey sequence shown in Table 2 [2], [3],[4].

Table 2. Farey Sequence

	0/1								1/0
Row 0	1/2								
Row 1	1/3	2/3							
Row 2	1/4	2/5	3/5	3/4					
Row 3	1/5	2/7	3/8	3/7	4/7	5/8	5/7	4/5	
	...								

In this table each rational number is generated from the two that brace it from above by adding numerators and adding denominators of this pair, e.g., 5/8 is braced by 3/5 and 2/3 so that 5/8 = (2 +3)/(5+3) and 5/7 is braced by 2/3 and 3/4. Beginning in row 0 and counting right to left, we find that 5/8 is the 10th rational in the hierarchy. Applying the above procedure, 10 corresponds to 1 0 1 00 (with last digit duplicated) which corresponds to the continued fraction [1,1,1,2] = 5/8. Continuing to the next row, you can check that 7/10 is indeed the 19th fraction in the hierarchy. It should also be noted that any term x in a row of the Farey sequence gives rise to two terms 1/(1+x) and x/(1+x) in the next row, e.g., x= 2/5 in row 2 gives rise to 5/7 and 2/7 in row 3.

Figure 1 illustrates the so-called devil's or satanic staircase generic to almost all dynamic systems [3]. This figure graphs the winding number ϖ vs a

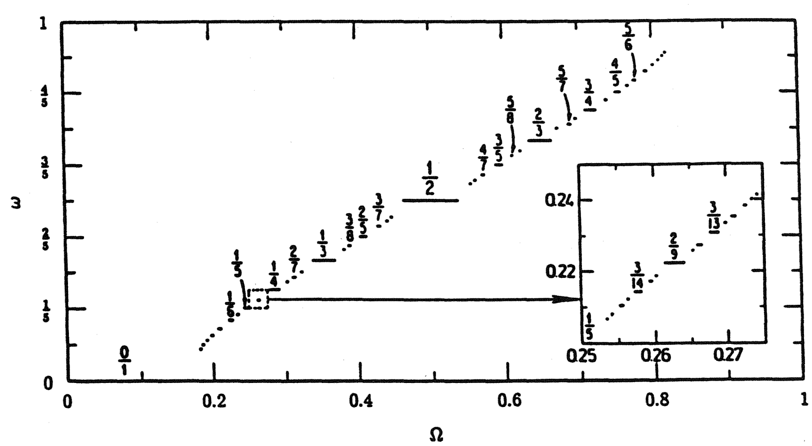

Figure1. Devil's staircase with plateaus at every rational number. From Fractals, Chaos and Power Laws by M. Schroeder. By permission of W.H. Freeman and Company.

frequency ratio Ω that represents the ratio of a driving force frequency and the resonance frequency of an oscillator for a system known as the circle map [3]. In this map a sequence of points z_0, z_1, z_2 ... are generated by the application of the function,

$$z_{j+1} = R(z_j) \text{ where } R(z) = z + \Omega - k/2\pi \sin 2\pi z.$$

The j indices can be thought of as time intervals. The winding number ϖ of the map is defined as the limit of $(z_n - z_0)/n$ as $n \to \infty$. Map R depends on a parameter k related to the energy of the system. As $k \to 1$, a critical value, the system approaches a chaotic state in which every winding number from the unit interval [0,1] is obtained depending on the value of Ω with rational values of ω *phase locked* to finite intervals of Ω. By phase locking we mean that the same winding numbers are manifested for a finite interval of Ω values. The phase locked intervals for the irrationals have zero width and so the irrationals form a kind of "dust" between the rationals. We see that the higher a rational number is in the hierarchy the wider is the phase locked interval corresponding to it. Winding numbers represented by larger intervals correspond to resonances of dynamical systems with greater stability justifying our reference to the rationals corresponding to these intervals as being "more important." For example the three widest plateaus occur for 1/2, 2/3, and 1/3 corresponding to the first three rationals of the hierarchy. The relative widths of the plateaus are ordered according to the terms of the Farey sequence with elements within each row having approximately the same width.

3. Gray Code, Divisibility, and the Towers of Hanoi

Notice that the strings of 0's and 1's (bit strings) in the third column of Table 1 are labeled as Gray code. Gray code is a system of representing integers as bit strings such that from integer to integer only a single bit changes in its representation unlike binary in which more than one bit can change from one integer to the next, e.g., 7 = 111 whereas 8= 1000. To go from one integer to the next in Gray code the value of the bit in the least significant digit changes (0 to 1 or 1 to 0) to give a bit string not already listed. The digits of the bit strings have been labeled 0,1,2,... from right to left and the digit of the Gray code that changes is listed in column 4; it is the sequence,

$$0102010301020104\ldots \qquad (1)$$

Where does this sequence come from? Take the integers and divide a given integer by the by the highest powers of 2 that goes evenly into it and record the exponent of the highest power which we refer to as the *index* of the factor (see Table 3). If an integer is not divisible by 2 then its index is 0.

Table 3. Divisibility of integers by powers of 2

Integer N :	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16 ...
Index :	0	1	0	2	0	1	0	3	0	1	0	2	0	1	0	4 ...

You will notice that sequence 1 has made its appearance again. Now if you remove the 0's, or alternatively, add 1 to each term, you get another familiar sequence,

$$121312141213121\ldots \qquad (2)$$

This corresponds to the sequence of moves required to solve the Towers of Hanoi puzzle described below. Next reduce each integer in this series by 1 to get 10201030102010... a replication of Series 1. Of course this transformation (remove the zeros and reduce by one each of the remaining numbers in Sequence 1) can be repeated ad infinitum so that we have found a self-similar pattern within the number system related to division by 2.

Note that if the modulus of any pair of adjacent numbers a/b and c/d in the Farey Sequence of Table 2 is defined to be $|ad-bc|$, then the Towers of Hanoi sequence is asymptotically developed for Row n as $n \to \infty$. For example, the sequence of moduli for Row 3 of Table 2 is

3537353 which corresponds to 1213121 with the following replacements: 3->1, 2->5, and 7->3. The moduli of the first 15 entries to the Farey sequence are shown in column 5 of Table 1.

The final idea in this cycle of ideas requires us to describe the Towers of Hanoi puzzle (abbreviated TOH). The Towers of Hanoi puzzle, an invention of the French Mathematician Edouard Lucas in 1883, is rich in number theoretic relations [2], [5], [6]. N disks of increasing

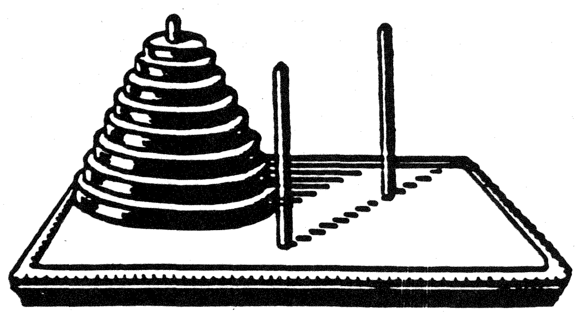

Figure 2. The Towers of Hanoi Puzzle with posts arranged in a circular fashion

sizes, numbered from 1,2,3,...,N from smallest largest are placed on one of three posts (see Figure 2). The object of the puzzle is to move all the disks to another post in such a manner that a large sized disk never lies atop a smaller sized disk during the transfer. If the posts are labeled A,B,C in a clockwise manner and the first move is in a clockwise direction, then the shortest path to the final configuration is unique. The sequence of moves is shown in column 8 of Table 1. Here the configuration

$d_1/d_2/d_3/\ldots d_m$ signifies that disk d_1 lies atop disk d_2 atop d_3 etc., e.g., 1/2/3 means that disk 1 lies atop 2 which lies atop 3.

An optimal transfer satisfies the following rules:

1) A smaller disk must always lie atop a larger disk;
2) The parity of two adjacent disks must also be different, i.e., adjacent disks must be even – odd or odd-even numbered, e.g., in the example, 1 lies on 2 and 2 lies on 3 in the above example.
3) During any transfer, an odd numbered disk moves clockwise (CW) while an even numbered disk moves counterclockwise (CCW).

Observe that the TOH positions are correlated with the integers. In fact the following algorithm enables one to convert from the binary representation of the integer to a TOH position:

Algorithm 2:

a) For a given integer, express its binary representation in terms of its number of repeated digits. For example, 22 = 1 0 11 0 is written as {1121} since the initial 1 and 0 and the final 0 are singletons whereas the middle 11 is a pair.

b) Starting at the right, this set of indices describes how to uniquely place n disks onto the three TOH posts. Starting on the right, the quantities of disks corresponding to the first index are placed on a post in the order 1,2,3,... The number of disks corresponding to the second index then go on another post.

c) The number of disks corresponding to the third index are then placed on a different post than the previous move. If a choice arises between an occupied post or a vacant post, always choose the occupied post. At all times the three rules stated above must be satisfied.

Following these rules for {1121} the TOH position is :

5 2/3 ¼

or, starting at the right, the first disk (#1) must go onto a post. Then the next two disks (#2 and #3) must go onto a different post. Then disk #4 must go onto a different post from the preceding one. However, according to rule c, we now have a choice:

Vacant post or occupied post

Always choose the occupied post as long as the parity rule is not violated. Therefore, #4 goes beneath #1. Then #5 must go onto a different post since it cannot go beneath disk #3 because of parity (no two odds together), so it must go onto the vacant post.

In this manner, each decimal number n is uniquely associated with a TOH position. Let us now make a short digression to convert a binary number directly to its equivalent Gray code representation. Consider 19, represented in binary as 10011. Multiplying by 2 and adding the next digit we get the sequence 1, 2, 4, 9, 19. These are the decimal representations, B_{19}, of the family of binary numbers: 1, 10, 100, 1001, 10011. Next take the differences of adjacent terms of the B_{19} sequence which we refer to as the Gray code sort,

n:	5	4	3	2	1
Binary:	1	0	0	1	1
B_{19}:	1	2	4	9	19
Gray code sort:	1	1	2	5	10

Taking each number of the Gray code sort mod 2 (i.e., even is 0 and odd is 1) yields the Gray code equivalent, 11010. However, perhaps more significantly, the Gray code sort indicates the number of times the nth disk moved, up to the Nth move in the transfer to the final TOH position, e.g., up to the 19^{th} move: disk 1 moves 10 times, while disks 2,3,4, and 5 move 5,2,1, and 1 times respectively.

Binary numbers can also be converted directly to Gray code by Algorithm 3.

Algorithm 3:

a) Record the leftmost digit of the binary representation.

b) If the k+1-th digit of binary is equal to or larger than the k-th digit then the k+1-th digit of the Gray code representation is the difference between these digits.

c) If the k+1-th digit is less than the k-th digit, then add 2 (base number of binary) to the k+1-th digit, and then subtract the k-th digit from it to obtain the k+1-th digit of Gray code.

4. The Generalized Towers of Hanoi Problem

We refer to the Towers of Hanoi puzzle as TOH:2 because, as we have seen, it is based on the binary numbers. We have been able to generalize this puzzle to the transfer of disks between n+1 posts which we refer to as TOH:n. Once again the n posts are arranged in a circular formation and, once again, we find that in the optimum transfer odd number disks move clockwise while even numbered disks move counterclockwise. Also the sequence of moves are related to the exponent of the greatest power of n, referred to as the index, that divides evenly into a given integer. Table 4 illustrates the divisibility sequence TOH:3.

Table 4. Divisibility by powers of 3

Integer N :	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	...
Index :	0	0	1	0	0	1	0	0	2	0	0	1	0	0	1	0	0	2	...

Eliminating the zeros results in the TOH:3 sequence 112112... It should be noted that in this sequence of moves, a disk is permitted to move either clockwise or counterclockwise from one post to an adjacent post. Thus if we wish to move a disk from one post to another clockwise from it by two post, we must transfer it in two steps.

We can also determine the Gray code sort corresponding to the n-ary, or base n, representation of the number of moves N and use this to predict the number of moves each sized disk makes during the optimal transfer. To do this express the TOH:n position in the number system base n. For example, step number 137 is represented in base 3 as $(12002)_3$. The sequence B_{137} in base 3 is then obtained by multiplying each digit of the binary representation by 3 and adding the next digit to get: 1,5,15,45,137 which gives the decimal values of the base 3 sequence: 1,12,120,1200, 12002. Next take difference to yield the Gray code sort:

n:	5	4	3	2	1
3-ary:	1	2	0	0	2
B_{137}:	1	5	15	45	137
Gray code sort:	1	4	10	30	92

Thus, disk 1 moves 92 times and disks 2,3,4,5 move 30, 10, 4, 1 time respectively. If the Gray code sort numbers are expressed mod 3 we get the generalized Gray code representation of 137 or 11100.

We can get the generalized Gray code of integer N directly from the base n representation by slightly modifying Algorithm 3 so that in step c the base number n is added to the k+1-th digit if it is less than the k-th digit. Just as for Gray code, a single digit changes between successive integers represented by generalized Gray code and the place value of the change numbered 0,1,2,...from the right represents the highest power of n that divides evenly into N. For example, expressing the following numbers in the base 4, 31 = $(133)_4$ and 32 = $(200)_4$ with Gray code representations, 31 = (120) and 32 = (220). Notice that a single digit changes in the third place (i.e., place value 2). Therefore 4^2 is highest power of 4 that divides evenly into 32. Also note that 31 mod 4 is gotten by adding the digits of the Gray code representation mod 4, e.g., 31 mod 4 = (1+2+0) mod 4 = 3. This holds for any integer N written in any base n. Table 5 lists the generalized Gray code for the first 64 numbers of the 4-ary system.

Table 5. 4-ary Gray Code and its Relationship to the DNA Codons

0	000	CCC	16	130	AUC	32	220	GGC	48	310	UAC
1	001	CCA	17	131	AUA	33	221	GGA	49	311	UAA
2	002	CCG	18	132	AUG	34	222	GGG	50	312	UAG
3	003	CCU	19	133	AUU	35	223	GGU	51	313	UAG
4	013	CAU	20	103	ACU	36	233	GUU	52	323	UGU
5	010	CAC	21	100	ACC	37	230	GUC	53	320	UGC
6	011	CAA	22	101	ACA	38	231	GUA	54	321	UGA
7	012	CAG	23	102	ACG	39	232	GUG	55	322	UGG
8	022	CGG	24	112	AAG	40	202	GCG	56	332	UUG
9	023	CGU	25	113	AAU	41	203	GCU	57	333	UUU
10	020	CGC	26	110	AAC	42	200	GCC	58	330	UUC
11	021	CGA	27	111	AAA	43	201	GCA	59	331	UUA
12	031	CUA	28	121	AGA	44	211	GAA	50	301	UCA
13	032	CUG	29	122	AGG	45	212	GAG	61	302	UCG
14	033	CUU	30	123	AGU	46	213	GAU	62	303	UCU
15	030	CUC	31	120	AGC	47	210	GAC	63	300	UCC

5. A Probability density function for divisibility of integers

We have previously shown that if each integer from the TOH:2 series , 12131214…, is reduced by 1 unit, the indices of Table 2 : 01020103… result. If the zeros are removed from this series, the TOH:2 series is replicated. This process can be repeated ad infinitum and also applied to the generalized TOH:n series to reveal the self-similarity of the TOH:n series. The following theorem is the result of this self-similarity process.

Theorem: The probability P that a randomly chosen integer is divisible by n^d but no higher power of n is given by,

$$p(n^d) = (1-x)x^d \qquad (3)$$

where x = 1/n.

Proof:

Since every n-th integer is divisible by n, the probability x that an integer is divisible by n is x = 1/n , and the probability that an integer is not divisible by n is 1-x. Therefore, the probability that an integer has index 0 (not divisible by n) in the TOH:n table corresponding to Table 4 is 1-x while the probability that it has a non-zero index (divisible by n) is x. By the self-similarity of the TOH:n series, the probability that an integer has index 1 (divisible by n^1 but no higher power of n) is (1-x)x, while the probability that the index is 2 or higher (divisible by all powers of n greater than 1) is x^2. That the probability of an integer having index d is $(1-x)x^d$ follows by induction.

Corollary: $1 = \sum_{d=0}^{\infty} p(n^d)$

Proof: Since $1/(1-x) = \sum_{d=0}^{\infty} x^d$,

$$1 = (1-x) \sum_{d=0}^{\infty} x^d \,.$$

The corollary follows from $p(n^d) = (1-x)x^d$.

Example 1: The probability that an integer is divisible by 3^2 but no higher power of 3 is $(1-1/3)(1/3)^2 = 2/27$. Therefore the number of integers smaller than 137 and divisible by 3^2 but no higher power of 3 is approximately: $(2/27)(137) = 10.148 \cong 10$ as derived above. The numbers are: 9,18,36,45,63,72,90,99,117,126. This result is approximate since the probability distribution applies asympotically as the integer approaches infinity.

The probability distribution function given by Equation 1 is :

$$1 = \overset{0}{(2/3)} + \overset{1}{(2/3)(1/3)} + \overset{2}{(2/3)(1/9)} + \overset{3}{(2/3)(1/27)} + \ldots$$

where the numbers above the terms in this equation refer to the probability distribution of $1,3,3^2,3^3,\ldots$e.g., the total number of integers less than 137 divisible by any power of 3 is the sum of terms 1,2,3, and 4 of this probability distribution multiplied by 137 or $(137)(2/3)(1/3+1/9+1/27+1/81) = 45.102 \approx 45$.

Example 2: The relative frequency of clockwise (CW) movements of the TOH:2 disks is gotten by adding the even terms of Equation 1, to get $1/(x+1)$, while the relative frequencies of the counterclockwise (CCW) terms are gotten from the odd terms of Equation 1, or $x/(1+x)$. For example, for TOH:2 $x = 1/2$, and two-thirds of the moves are CW while one-third are CCW, a ratio of CW to CCW movements of 2:1. Note that $1/(1+x)$ and $x/(1+x)$ are the pair of terms of the Farey series resulting from the fraction $x = 1/n$.

6. Relationship between 2-ary Gray Code and the Logistic Equation of Dynamical Systems Theory

The logistic equation of dynamical systems theory [2] is given by,

$$x_{n+1} = a\, x_n(1-x_n) \qquad (4)$$

Given a sufficiently small value of 'a' and beginning with x_0 on the interval [0,1], the successive iterates (considered to be time intervals) of Equation 4 constitute a *trajectory* on [0,1]. It is well known that as values of 'a' are increased to the critical value of 3.5699..., the trajectories approach periodic orbits of periods 2^n for n = 1,2,3,... For values of 'a' exceeding the critical value, the system enters the realm of 'chaos' in which small changes in the initial conditions result in wildly different trajectories given a sufficient number of iterates (time). If we label values of x_n by 0 if they lie either at the maximum point or to the left of the maximum point of the logistic function y = ax(1-x) , and 1 if they lie to the right of the maximum, the resulting string of 0's and 1's is known as the *symbolic dynamics* of the logistic equation.

The symbolic dynamics of the logistic equation can be generated from the binary sequence of integers in two steps:

1. Add the digits of the binary numbers mod 2 to get the so-called Morse-Thue sequence [Schroeder 1991] , e.g,
   0, 1, 10, 11, 100, 101, 110, 111, 1000, 1001, 1010, … → 0 1 1 0 1 0 0 1 1 0 0 …
2. The symbolic dynamics is gotten by transforming the Morse-Thue sequence to Gray Code by Algorithm 3. This is equivalent to adding adjacent binary digits mod 2 to get what we refer to as the *Complementary Morse-Thue sequence* (CMT).
   1 0 1 1 1 0 1 0 1 0 …

Notice that the symbolic dynamics of the CMT sequence is equivalent to the parity of the Towers of Hanoi sequence 121312141213121… where even digits are assigned 0 and odd digits 1 (i.e., CW = 1 and CCW = 0). Therefore, the 1's and 0's of CMT are governed by the probability density function of Equation 3, i.e., the ratio of 1's to 0's is 2:1.

7. The Relationship between 4-ary Gray Code and DNA

It is well known that DNA is composed of 64 codons. Each codon is a three letter "word" made up of one of four bases C (Cytosine), A (Adenine), G (Guanine), U/T (Uracil/Thymine). There is a natural way to relate the DNA codons to Gray Code. Use the following assignments:

C= 0, G = 1, A = 0, U/T = 1 .
0 1 1 0

For example CUG = 011 and GAC = 100. GAC is called the anti-codon of CUG since
001 110
1 of the codon is replaced by 0 and 0 by 1 of the codon to get the anti-codon. Notice that the upper and lower bit strings of both the codon and anti-codon differ in a single bit, e.g., they have a Hamming distance of 1. Subtracting the Hamming distance from 9 yields the number of hydrogen bonds per codon/anti-codon pair [7]. For example, both CUG and GAC have 9 – 1 = 8 hydrogen bonds.

In Table 6 the codons are arranged in an 8x8 square pattern along with their number of hydrogen bonds. In this square both the row and column numbers are labeled 0 to 7 in the standard Gray Code, e.g., 000, 001, 011, 010, 110, 111, 101, 100, and each element of the table is listed by a 6-bit representation. This is equivalent to a Karnaugh map for a Boolean system with six variables. The Karnaugh map is a commonly used tool to simplify compound statements in Boolean (2-valued) logic [8]. In this table, as in all Karnaugh maps, adjacent elements to the left, right, up, down, or wrap-around of any element differs from that element in a single bit. Also to find the location of an anti-codon given the position (row and column) of a codon, or vice versa, use the following algorithm: row (column) 0 matches row (column) 5, 1 matches 4, 2 matches 7, 3 matches 6. For example, GGU is located in row R_4 and column C_5 which becomes R_1 and C_0 for the anti-codon CCA and each have 8 hydrogen bonds.

Table 6. Gray Code DNA Matrix

	0	1	2	3	4	5	6	7
	000	001	011	010	110	111	101	100
0	000	000	000	000	000	000	000	000
	CCC 9	CCU 8	CUU 7	CUC 8	UUC 7	UUU 6	UCU 7	UCC 8
	000	001	011	010	110	110	101	100
1	001	001	001	001	001	001	001	001
	CCA 8	CCG 9	CUG 8	CUA 7	UUA 6	UUG 7	UCG 8	UCA 7
	000	001	011	010	110	111	101	100
2	011	011	011	011	011	011	011	011
	CAA 7	CAG 8	CGG 9	CGA 8	UGA 7	UGG 8	UAG 7	UAA 6
	000	001	011	010	110	111	101	100
3	010	010	010	010	010	010	010	010
	CAC 8	CAU 7	CGU 8	CGU 9	UGC 8	UGU 7	UAU 6	UAC 7
	000	001	011	010	110	111	101	100
4	110	110	110	110	110	110	110	110
	AAC 7	AAU 6	A GU 7	AGC 8	GGC 9	GGU 8	GAU 7	GAC 8
	000	001	011	010	110	111	101	100
5	111	111	111	111	111	111	111	111
	AAA 6	AAG 7	AGG 8	AGA 7	GGA 8	GGG 9	GAG 8	GAA 7
	000	001	011	010	110	111	101	100
6	101	101	101	101	101	101	101	101
	ACA 7	ACG 8	AUG 7	AUA 6	GUA 7	GUG 8	GCG 9	GCA 8
	000	001	011	010	110	111	101	100
7	100	100	100	100	100	100	100	100
	ACC 8	ACU 7	AUU 6	AUC 7	GUC 8	GUU 8	GCU 8	GCC 9

It has been found that the amino acids are formed from contiguous groups of codons, e.g., proline: CCC, CCU, CCA, CCG; glutamine: CAA, CAG; leucine: CUU, CUC, CUG, CUA, UUA, UUG; etc. [7]. Apparently Gray code arises in genetics as a means of minimizing the "cliffs" or mismatches between the digits encoding adjacent bases and therefore the degree of mutation or differences between nearby chromosome segments. The requirement in an encoding scheme is that changing one bit in the segment of the chromosome should cause that segment to map to an element which is adjacent to the premutated element.

There is a natural relationship between 4-ary Gray Code and the DNA codons which can be understood by making the following correspondences: 0→C,

1→A, 2→ G, 3→ U/T in Table 5 and C → 0, A→ 0, G → 1, U/T = → 1 in Table 6.
0 1 1 0

This enables Table 6 to be rewritten as Table 7 using the 4-ary Gray code in Table 5. The corresponding decimal values are listed in Table 8. For example, the bit string in Row 3, Column 7 of Table 6 is 101 which corresponds to UAG or Gray Code 312 according
011
to Table 7 or decimal 50 according to Table 8. Notice that Table 7 inherits the property that each codon differs from an adjacent codon: up, down, right, left, wrap-around, in a single bit. Table 7 reveals the 4-ary number system as the natural system with which to characterize the DNA codons. The integers from 0 to 63 are divided into four quadrants. Each quadrant is subdivided into four compartments of four codons.

Table 7. 4-ary Gray Code Matrix

000	003	033	030	330	333	303	300
001	002	032	031	331	332	302	301
011	012	022	021	321	322	312	311
010	013	023	020	320	323	313	310
110	113	123	120	220	223	213	210
111	112	122	121	221	222	212	211
101	102	132	131	231	232	202	201
100	103	133	130	230	233	203	200

Table 8. Decimal Values for 4-ary Gray Code Matrix

0	3	14	15	58	57	62	63
1	2	13	12	59	56	61	60
6	7	8	11	54	55	50	49
5	4	9	10	53	52	51	48
26	25	30	31	32	35	46	47
27	24	29	28	33	34	45	44
22	23	18	17	38	39	40	43
21	20	19	16	37	36	41	42

8. Conclusion

Galileo stated that, "Nature's great book is written in mathematical symbols." We have shown that we can uncover the secrets of number by holding it up to the light in the proper way. Information about naturally occurring resonances, the nature of dynamical systems, and the structure of DNA are already built into the number system through a natural hierarchy of rational numbers. The Farey sequence and continued fractions were introduced as tools to facilitate an understanding of this hierarchy. We have also shown that the structure of the integers is isomorphic to the Towers of Hanoi puzzle and its generalizations. Gray code and its generalizations was introduced as an aid to understand this puzzle and also as the natural framework for systematizing the 64 codons of DNA.

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