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Mathematical Connections
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Musical Composition as Applied Mathematics: Set Theory and Probability in Iannis Xenakis's *Herma*

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Abstract

Originally trained as an architect and engineer, Greek composer Iannis Xenakis has forged a distinctive musical style through the application of various branches of mathematics to his compositional process. Early in his career he focused on topics in probability. After his initial concentration on mathematical models of indeterminacy, he began to incorporate more deterministic methods of organization into his compositions, including set theory and group theory. *Herma* is a composition for solo piano written in 1960-61. It is representative of the coexistence of determinacy and indeterminacy in the structure of his works, and thus serves as a good introduction to the music of this remarkable composer.

1. Xenakis and the Development of Stochastic Music

1.1 Xenakis the Composer. The modern period is an era in which progressive composers have been both lauded and repudiated for their use of abstract compositional procedures. Even among his contemporaries Iannis Xenakis stands out for the direct and thorough manner in which he has applied procedures from various branches of mathematics to the composition of his music. His most important theoretical treatise, *Formalized Music: Thought and Mathematics in Music*, presents a variety of mathematical concepts along with their applications to his compositions over roughly a forty-year period [1]. In the 1950s his work was mainly concerned with the application of probability theory to musical composition, resulting in a style he dubbed "stochastic music." Beginning in the mid-1960s his work focused on formalized methods of deterministic (i.e., non-stochastic) composition including a generalized theory of musical scales. His subsequent work has been less formalized and more eclectic, including the use of freehand graphic designs in the formation of musical structures. In the 1990s, however, he has demonstrated a desire to achieve a pure form of stochastic music in the electronic domain through the creation of works in which sound waves themselves are subjected to stochastic transformations.

Xenakis came rather late in life to composition. He was born in 1922 in Romania to parents of Greek origin and received a conventional Greek education before pursuing studies in mathematics, engineering and architecture at Athens Polytechnic. His education was interrupted by the political turmoil of World War II. He involved himself in the Greek Resistance movement, was incarcerated several times and was eventually blacklisted. His father arranged for him to escape from Greece in 1947. He settled in France, where he found work in the architectural firm of Le Corbusier. Soon after his arrival in France Xenakis became interested in composition. Le Corbusier was knowledgeable in music, and through him Xenakis made contacts in the world of the French post-war avant-garde. Chief among these contacts was Olivier Messiaen, France's leading composer at the time. Messiaen accepted Xenakis as a pupil, despite his unconventional background and minimal musical training. During the early 1950s he had the opportunity to study alongside Messiaen's most famous pupils, Pierre Boulez and Karlheinz Stockhausen. Now a French citizen, Xenakis is regarded, along with Boulez, as one of France's most important senior composers.

According to his principal biographer, Xenakis turned to music partly as a form of therapy, a way of coming to terms with the traumatic events he had experienced in Greece during and after the war [2]. The expressive urge that underlies his music coexists in an uneasy alliance with his desire to create robust and enduring structures, a legacy no doubt of his architectural training. Vying with one another in Xenakis's music is the Greece of philosophers and mathematicians and the Greece of horrifying tragedies, spectacular battles, and persistent political conflicts. The result is a music that is expressive, yet shorn of all sentimentality: a music that exalts the life of the mind while stimulating the senses with uncompromising, even confrontational, intensity.

1.2 The Development of Stochastic Music. The musical world into which Xenakis entered in post-war France was a world in reconstruction, like much of European society at the time. Many composers felt a need to resume the progressive trends in the arts that had been threatened by fascist intrusions into the cultural life of Western Europe. Young composers wanted to start over again from ground zero, hoping to leave behind the retrogressive conventions that had been encroaching on modernism since the 1920s and 1930s. Stravinsky's neoclassicism was seen as a decline and a betrayal of the progressive cause, while the work of Schoenberg and Webern—which had been banned by the Nazis—seemed to point the way forward. Young composers, including Boulez and Stockhausen, began to generalize the serial organization of pitches found in Schoenberg and Webern to other aspects of musical structure, including rhythm, form, and even instrumentation. Messiaen himself did some work along these lines as well. Paradoxically, the stricter the organization of the various elements of musical sound was, the more likely the music was to be perceived as a manifestation of disorder. Xenakis saw a contradiction here between ends and means and sought a logical solution: the deliberate composition of disorder through the application of probability theory. He dubbed his new method "stochastic music."

Herma, a seven-minute composition for piano solo written in 1960-61, is the most concentrated example of Xenakis's stochastic music. Unlike Xenakis's first stochastic compositions, which were written for orchestra, chamber ensembles or electroacoustic media, *Herma* is his first stochastic work for a solo instrument. *Herma* is also a transitional work, for while the sections of the piece consist of "clouds" of pitches that are distributed according to specific randomizing functions, the selection of pitches follows a deterministic process based on set theory. *Herma* thus combines indeterminacy—manifested as stochastic processes at the local level—with determinacy—manifested as a sequence of set-theoretic operations at a global structural level. Probability also plays a role in the work's global temporal structure, which is discussed in the concluding section of this paper.

2. Set Theory as a Basis for Large-Scale Pitch Structure

The large-scale pitch structure of *Herma* is organized according to the principles of set theory. Xenakis begins by defining the 88 pitches available on the standard piano keyboard as the universal set. Using slightly non-standard terminology, he refers to this as the referential set, R. Three smaller sets—A, B, and C—were then constructed from the elements in R. Each of these sets contains about one-third of the pitches in R. The contents of sets A, B and C overlap slightly, ensuring that there will be no trivial, i.e. empty, intersections among the three sets. The standard set-theoretic operations of union, intersection and complementation form the basis for the selection of pitches in the sections of *Herma*.

Xenakis lends structure to the set-theoretic procedures by defining two sequences of operations that lead to an identical final set, F, whose contents are represented by the Venn diagram in Figure 1. Two distinct but equivalent formulas define the contents of set F:

$$ABC + \overline{AB}C + A\overline{BC} + \overline{A}B\overline{C} \qquad \text{and} \qquad (AB + \overline{AB})C + \overline{(AB + \overline{AB})}\,\overline{C}$$

The formula on the left is the result of a sequence of seventeen operations performed on the primary sets A, B and C. The more concise formula on the right is the result of a sequence of ten operations performed on the primary sets. Xenakis interprets the sequences of operations as competing paths toward a common goal, thus introducing an element of structural drama into the presentation of pitches as the work unfolds in time.

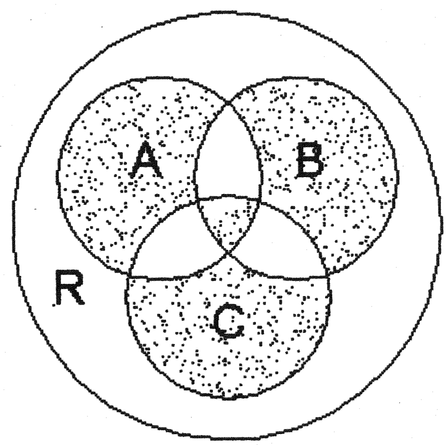


Figure 1: *Venn diagram of set F*

In his explanation of the compositional process used in *Herma*, Xenakis uses flow charts to specify the sequences of set-theoretic operations that result in set F [3]. In the finished composition, however, the order of operations in both sequences has been rearranged, some operations are omitted, and some operations appear that do not belong to either sequence. There is no mathematical justification for these modifications, but from a musical perspective the shape of the finished composition makes a good deal of sense. Figure 2 shows a structural plan for *Herma*, summarizing the labeling of sets and other information found in the musical score [4]. The work begins with an introductory section that presents the pitches in R, the referential set. This is followed by the presentation of the primary sets, each one followed directly by its complement. The presentation of the sets up to this point—at about 4', approximately three-fifths of the way through the piece—follows the musical logic of thematic presentation, in which the basic materials of a work are presented in a clearly audible sequence. The remainder of *Herma* features a musical "development" of the primary sets. Thematic development in music consists of procedures that result in the transformation of themes. In traditional music these procedures include the fragmentation of themes into smaller musical units called motives, the combination of motives from different themes, and a general increase in harmonic and rhythmic activity. In *Herma* the process of development consists of the presentation of intersections involving the primary sets and their complements. These subsets are presented in sections that are generally shorter than those in which the primary sets and their complements were first introduced. The latter part of the work also features the repetition (or "recall," to use Xenakis's term) of several of the subsets following their initial appearances.

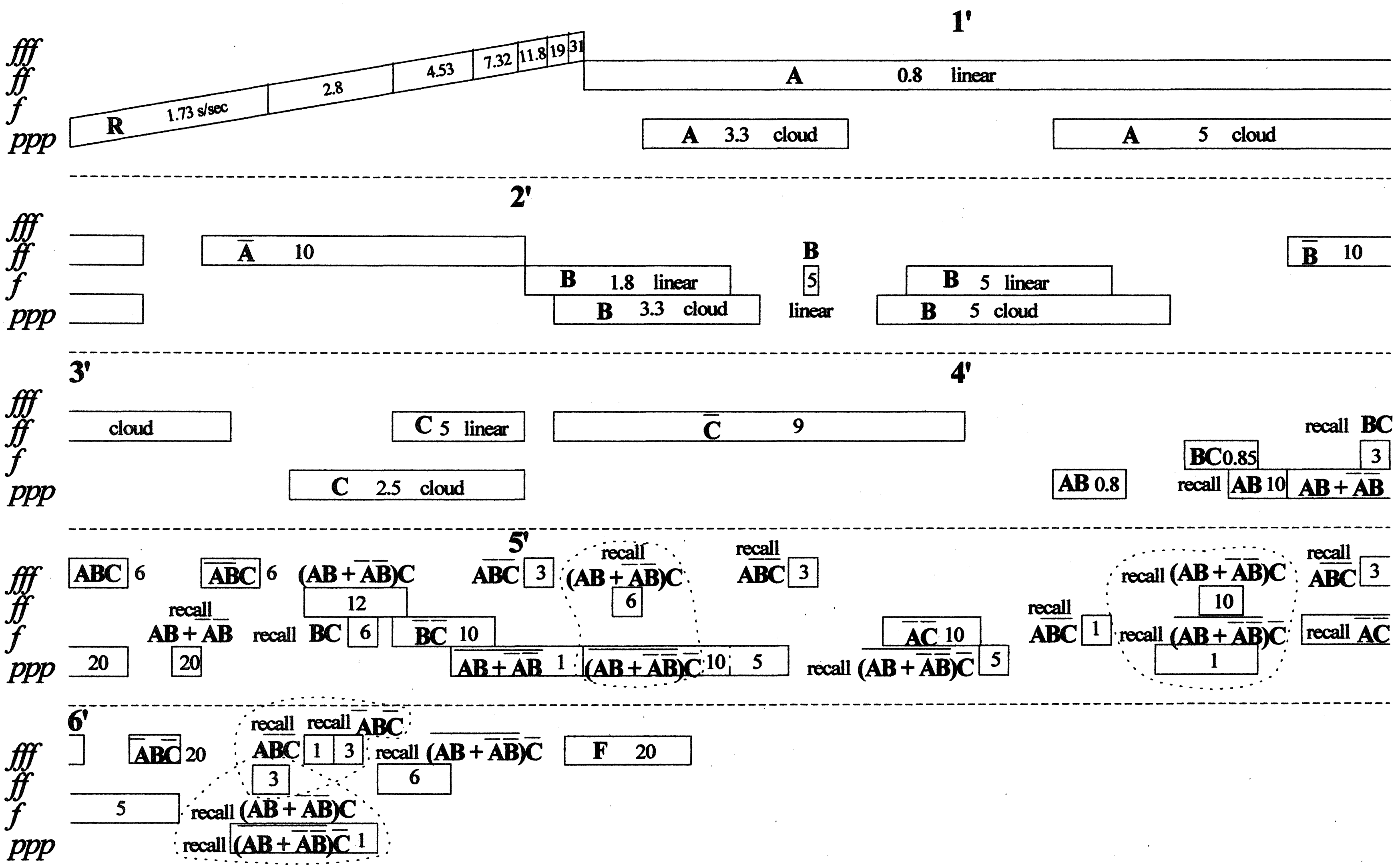


Figure 2: *Structural plan for* Herma

Variety is maintained on the musical surface through changes in the pitch contents of the sets and also through changes in the other characteristics that are represented in Figure 2. One of these characteristics is the loudness at which the pitches in the sets are to be played. Four levels of loudness are used in *Herma*: *pianisissimo* (*ppp*), extremely soft; *forte* (*f*), loud; *fortissimo* (*ff*), very loud; and *fortisissimo* (*fff*), extremely loud. The introduction begins at the lowest of these levels and moves progressively to the highest. The rest of the sections remain at a constant level of loudness throughout. Another characteristic shown in the chart is the density of the sections. Xenakis defines density in his stochastic as music as the average number of sounds that occur per second of music. The density increases within each consecutive subsection of the introduction while the lengths of the subsections become progressively shorter. The combination of all of these factors results in an increase in musical activity that reaches a fever pitch just before the presentation of set A. In the first part of the work most of the sections bear the descriptive terms "linear" and "cloud." In the three overlapping sections in which set A is presented, these terms appear to describe the effect of the relative density of each section: the least dense section (at 0.8 sound/sec) sounds like a linear succession of notes, while the denser sections give the impression of clouds of notes. The cloud effect is enhanced by the use of the damper pedal in these sections, which allows the resonance of the notes to continue after the keys are released. The descriptive terms do not correlate as nicely with the density or the pedaling effects in the remaining sections, however, so the reason for their inclusion in the score is not entirely clear.

The frequent overlapping of sets is a third characteristic that lends considerable variety to the musical surface. Whenever sets overlap the effect on the density of the musical surface is cumulative. Overlap also causes pitches from sets at different levels of loudness to be brought into close proximity with one another. This serves to differentiate between the contents of the sets, while the combination of different levels of loudness results in a colorful and varied musical surface. In the latter portion of the piece, the overlapping of sets allows for the gradual assemblage of the elements that combine to form set F. The contents of set F are foreshadowed several times prior to its definitive presentation at the end of the work, where it is given at the highest level of loudness and at a very high density of 20 sounds/sec. The contents of F are first foreshadowed just after 5 minutes and again just before and after 6 minutes. Each of these foreshadowings is circled with a dotted line in Figure 2. This is another example of Xenakis's adaptation of set-theoretic logic to musical purposes, for the foreshadowing of important themes is a frequent technique in classical music, especially in music written since the mid-19th century.

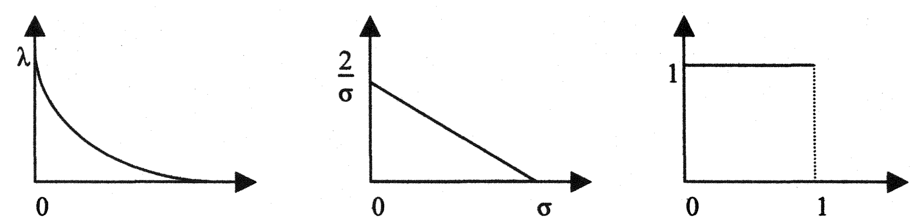
3. Stochastic Composition at the Local Level

The previous section presented an overview of the global pitch structure of *Herma* with respect to the three primary sets and their transformations over the course of the work. I turn my attention now to the stochastic aspects of the work's structure. This occurs at two levels: first, the composition of the pitch and rhythmic structure of the sections (represented by boxes in Figure 2); and second, the composition of the lengths of the sections and of their placement within the total time span of the work. The first of these levels will be dealt with here. The second level will be discussed in the following section.

The fundamental principle underlying Xenakis's stochastic music is the application of probability theory to the process of musical composition. The unit of structure to which probability theory is applied in this music is the interval. The interval is a musical term for the distance between two sounds with reference to some characteristic of those sounds. One may speak, for example, of an interval between two pitches, between the times at which two sounds begin, or between two degrees of loudness. The sounds of stochastic music result from the generation of pseudo-random interval successions for two or more characteristics of sound. The two most basic characteristics are pitch and the time at which a sound begins, known as its "attack time." Intervals for each characteristic are calculated independently and the endpoints of the intervals are brought together to determine the pitches and attack times of individual sounds. Each sound may therefore be thought of as a point in a multidimensional space whose coordinates are determined by pseudo-random interval successions operating independently in each dimension.

Xenakis has chosen which probability distributions to apply to which characteristics of sound by determining which distributions produce the most appropriate musical effect [5]. For the composition of the intervals between attack times he has invariably used the exponential distribution. In probability theory this distribution is used to model the waiting times between relatively rare phenomena such as automobile accidents in a city or the appearance of meteors in the night sky [6]. Xenakis has used several different distributions, however, to determine the intervals between pitches in a set. The simplest of these is the linear distribution. Since the linear distribution contains only positive values, it is useful for determining the size but not the direction of the pitch intervals. Another distribution, such as the uniform distribution, must be used in conjunction with it in order to determine whether the intervals move up or down. A situation can be set up in which there is a 50/50 chance that the interval will be given a positive sign (i.e., it will go up), or a negative sign (i.e., it will go down).

The general characteristics of the exponential, linear and uniform distributions are shown in Figure 3. All three are continuous distributions, meaning that intervals of any size may occur within the range of values defined for each distribution. Sample graphs representing the probability density functions of each distribution are shown at the top of the figure. The graph for the exponential distribution shows that, as the size of an interval x increases linearly, the probability of its occurrence decreases exponentially. The parameter λ represents the average interval of time between events in the exponential distribution. The graph of the linear distribution shows that, as the size of an interval x increases linearly, the probability of its occurrence decreases linearly. The parameter σ represents the upper limit of the interval size within the linear distribution. Finally, the graph of the uniform distribution shows that, as the size of an interval x increases linearly, the probability of its occurrence remains constant.



distribution:	exponential	linear	uniform
probability density function:	$f(x) = \lambda e^{-\lambda x}$, for $x \geq 0$	$f(x) = \frac{2}{\sigma}\left(1 - \frac{x}{\sigma}\right)$, for $0 \leq x \leq \sigma$	$f(x) = 1$, for $0 \leq x \leq 1$
cumulative distribution function (CDF):	$F(x) = \int_0^x f(x)\,dx = 1 - e^{-\lambda x}$	$F(x) = \int_0^x f(x)\,dx = \frac{x}{\sigma}\left(2 - \frac{x}{\sigma}\right)$	$F(x) = \int_0^x f(x)\,dx = x$
inverse of CDF:	$F^{-1}(u) = \frac{-\ln(1-u)}{\lambda}$, for $0 \leq u \leq 1$	$F^{-1}(u) = \sigma\left(1 - \sqrt{1-u}\right)$, for $0 \leq u \leq 1$	$F^{-1}(u) = x$, for $0 \leq u \leq 1$

Figure 3: *Properties of probability distributions*

The cumulative distribution function (CDF), $F(x)$, is found directly below the probability density function, $f(x)$, for each distribution in Figure 3. $F(x)$ is the integral of $f(x)$ from 0 to x within the range of possible values for x. As the integral of $f(x)$, $F(x)$ may be represented as the area beneath the curve of $f(x)$. For a given distribution, the CDF defines the probability of finding an interval whose size is somewhere between 0 and x. As x approaches its upper limit, the probability of finding an interval whose size is between 0 and x approaches 1. The probability thus varies from 0—absolutely no probability of finding such an interval—to 1—absolute certainty of finding such an interval—as the value of x increases from 0 to its upper limit.

Two methods may be used to produce successions of intervals whose statistical properties approximate the properties of specific probability distributions. In the first method, one begins by dividing the range of possible values for x into equally sized segments. In this way it becomes possible to define the probability of the occurrence of intervals whose sizes fall within specific ranges. By choosing an appropriate number of intervals from within each range and placing them end to end, one may approximate the effect of a random succession of intervals. In addition, the statistical characteristics of this interval succession will resemble those of the probability distribution with which one began. The interval succession may also be subjected to statistical tests to measure the degree of its resemblance to true randomness. Composition by this method is a slow and laborious procedure, but it was the only method available to Xenakis when he composed his first stochastic works in the early-to-mid 1950s.

The second method involves automated computation. In this method, values are drawn from a computer's pseudo-random number generator and are fed through mathematical formulas derived from probability distributions. This process results in interval successions whose statistical characteristics resemble those of the probability distributions from which the formulas were derived. This approach depends on the fact that the CDF of a probability distribution associates each interval within the range of 0 to x with a unique probability whose value lies between 0 and 1. This association works in the opposite direction also, so that one may begin with a specific probability value in order to find the unique interval size with which it is paired in the distribution's CDF. When the computer's pseudo-random number generator is used to determine the probability values, the result is a pseudo-random interval succession whose statistical properties approximate those of the desired probability distribution. The formulas used to produce interval successions that approximate the statistical properties of the exponential, linear and uniform distributions are shown in the bottom row of the table in Figure 3. The variable u in each formula represents a pseudo-random probability value between 0 and 1. For every value of u there is a corresponding value of x which is found by feeding u through the inverse of the CDF of the probability distribution, i.e. $F^{-1}(u)$.

Xenakis developed and executed a computer program in Fortran for the composition of stochastic music between 1956 and 1962. The works composed with that program were first performed beginning in 1962. Given that *Herma* was composed in 1960-61, it is possible that at least part of it may have been composed with the aid of the computer program, but it is also possible that the entire work may have been written using the original method of calculation by hand.

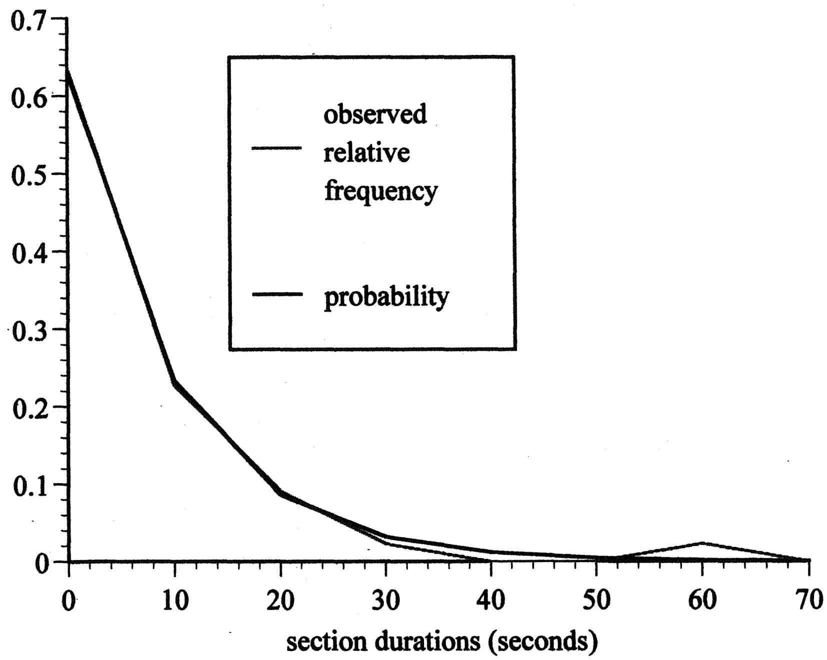
4. The Large-Scale Temporal Structure of *Herma*

The two methods of stochastic music outlined above may at first seem abstract, perhaps even obscure. Once one has gotten hold of the necessary formulas and understands how to implement them musically, however, both methods are actually quite straightforward and easy to use. In the descriptions of the derivation of the pitch sets and the methods of stochastic composition, my presentation has followed Xenakis's own account of these procedures quite closely. I would now like to turn the investigation of *Herma* in a more speculative direction by using the principles of stochastic composition as the basis for an analysis of the work's large-scale temporal structure. Several authors have written about *Herma*, some devoting their attention mainly to its pitch structure, but no one has yet demonstrated an understanding of how its large-scale temporal structure results from the application of probability theory [7]. The tendency to dwell on the work's pitch structure is unfortunate, since the effect of the music has much more to do with its large-scale temporal structure than with the specific contents of its sets. The composer appears to have thought so as well, for the published score remains faithful to the structural plan in Figure 2 in terms of the lengths of the sections and the densities within the sections. There are, however, irreconcilable differences between the contents of the primary pitch sets—A, B, and C—and the contents of the sets derived from them.

The first thing that one is likely notice while observing the lengths of the sections in Figure 2 is that their arrangement does not appear to be completely random. While one very short section is included in the first part of the work (set B, ca. 2'30"), the majority of the short sections are found in the work's second part, beginning with set AB (after 4'). The separation of the long sets from the short ones appears to have been done by design, for the time at which set AB begins—4'07" (247")—is approximately the golden section of the work's total duration (6'44" [404"]): 247"/404" ≈ 0.611. (The golden section ≈ 0.618.) The golden section, of course, is a classic proportional division. It forms the basis of Le Corbusier's approach to architectural design, as explained in two books on the Modulor, a proportional system of measurement that he invented [8]. Xenakis's first published essay, in fact, is the appendix to the second volume on the Modulor. In this essay, Xenakis explains how Le Corbusier's ideas regarding the structuring of architectural space were applied to the temporal structure of his first orchestral work, *Metastaseis* (1953-54). In *Herma*, the use of the golden section creates a dynamic, asymmetrical balance between the introduction of the primary sets and their complements in the first portion of the work and the transformations of these sets in the second portion.

While the division of the work into two parts according to the golden section is clear, the reason for the different lengths of the work's individual sections may not be so apparent. In point of fact, the distribution of lengths among the sections closely resembles an exponential distribution. This can be demonstrated by first removing the sections from their immediate temporal context. From this perspective, the sections may be viewed solely in terms of their duration. The section durations may be sorted by length and their distribution within the work as a whole may be compared an ideal exponential distribution. The sum of the section durations in the work is 7'22" (442"), which is 38" longer than the work's performance duration. (This discrepancy results from the overlap of several sections during the course of the work. It should also be noted that the periods of silence intervening between the sections are not taken into account in this calculation.) There are 44 sections in all. Dividing the number of sections by the sum of their durations gives an average duration of 10 seconds per section. In terms of events per second, the average section duration of 10 seconds may be expressed as 0.1 sections per second. The actual distribution of section lengths, which constitutes the observed sample, may thus be compared with an exponential distribution in which $\lambda = 0.1$. A comparison of the observed relative frequencies and their probabilities according to the exponential distribution is shown in Figure 4a. As the graph in the figure demonstrates, the two distributions are very similar.

The intervals between the start times of the sections also approximate an exponential distribution. The intervals between start times are measured with respect to the work's actual temporal structure. When the interval between the start times of two sections is smaller than the length of one of the sections, the sections overlap. An example of this is the group of sections that that present the pitches belonging to set A (see Figure 2). When the interval between start times is larger than the length of the first of the two sections, a measured period of silence results. This occurs, for example, between the presentation of set A and the presentation of its complement. The number of sections is, once again, 44, but the performance duration of the work is 6'44" (404"), which is 38" shorter than the sum of the durations of the sections. The average rate of occurrence of the sections is 44/404" ≈ 0.11 sections per second. A comparison of the a.



b.

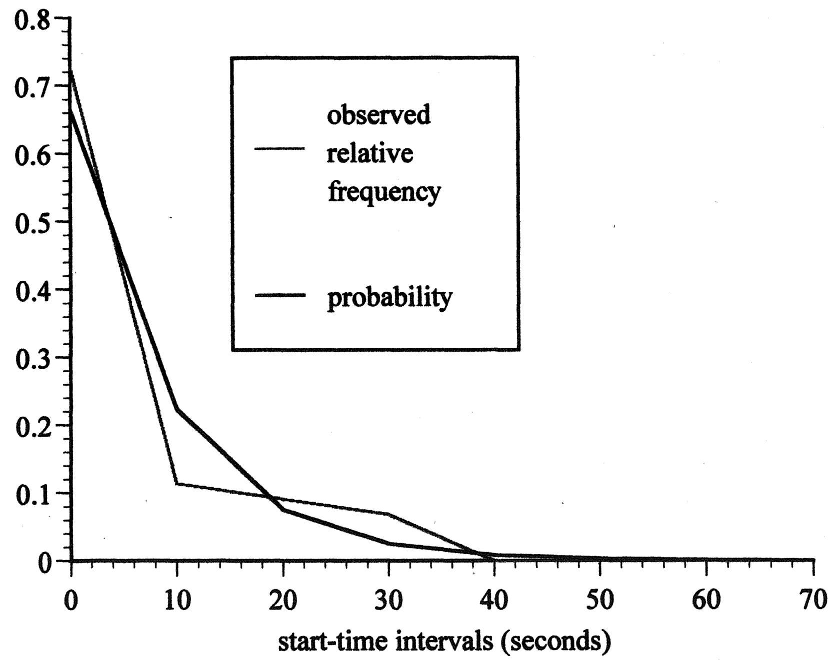


Figure 4: *Distribution of sections outside-time and in-time*

observed relative frequency of start-time intervals with an exponential distribution in which $\lambda = 0.11$ is shown in Fig. 4b. The two distributions in this case are not as close as in Fig. 4a, but nonetheless there is a noticeable resemblance between the observed relative frequency of start-time intervals and the probabilities defined by the exponential distribution.

Xenakis's apparent use of the exponential distribution in determining the lengths of the sections as well as their start times results in a unification of the local and global temporal structures in *Herma*. On the local level, as indicated in the previous section, the exponential distribution was used to determine the intervals between the attack times of the individual notes within the sections of music. On the global level, the same distribution was apparently used to determine the lengths of the sections and the arrangement of sections within the temporal structure of the finished composition. In his theory of musical time, Xenakis draws a distinction between duration as a property of a musical sound and the time at which that duration is set to begin in an actual composition. Any of the properties of individual sounds—including pitch, instrumentation, loudness, and duration—may be determined independently of the eventual placement of those sounds in a compositional context. Although duration depends upon the existence of time as a quantifiable phenomenon for its measurement, this time may be regarded as an abstract potential apart from its actualization in a concrete musical context. Thus, all of the properties of individual sounds specified above may be regarded as "outside-time" characteristics of musical sounds. The combination of the outside-time characteristics of sounds with their locations in a concrete musical context results in the "in-time" structure of a musical composition [3]. Extending Xenakis's theory to include not only individual sounds but also clearly delineated sections of music, it is clear from the illustrations in Figure 4 that both the outside-time and in-time structures of the sections—that is, both their lengths and their placement within the temporal flow of the finished composition—approximate the exponential distribution. The organization of the section lengths, however, seems not to have been randomized (as were the attack times of the individual notes within sections), but rather is indicative of the composer's will to place most of the long sections before the work's golden section and most of the short sections after it.

5. Concluding Remarks

Xenakis presents a special case of the thinking musician. Unlike other musicians, such as Arnold Schoenberg and Milton Babbitt, who used abstract thought to codify technical advances that grew directly out of previous musical tradition, Xenakis conceived of abstract structures first and then sought for ways to realize these structures in sound. The result is a bold and original approach to composition that favors explosive energy over subtle nuance, objectivity over sentimental expression, and logical principle over spontaneous improvisation. Xenakis is a builder of musical structures, an architect of volatile sound textures. His tools are the basic elements of abstract thought as expressed in mathematics. In *Herma* the specific tools used are set theory, representing deterministic logic, and probability, representing the logic of indeterminacy. Together these opposing forms of logic work to create a dialectical structure that is both dramatic and ferociously beautiful to those who can withstand the intensity of its expression. Even in this, one of the composer's most conceptually pure compositions, mathematical logic is used as a tool for the development of clearly articulated musical structures. The structure of the music is not simply the result of mathematical procedures. Rather, mathematics has been used in the service of the intended musical effect. Through the development of his unique brand of applied mathematics, Xenakis has managed to forge a distinctively original style and to infuse a renewed sense of vigor and energy into the musical avant-garde of the late twentieth century.

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