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The Subtle Symmetry of Golden Spirals

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Abstract

The beautiful symmetries of golden spirals are intuitively evident to everyone that sees them even for the first time. The purpose here is to show how these symmetries are manifestations of the special nature of the eyes of those spirals.

1. The Golden Spiral Circumscribing Golden Triangles

Following the ancient Greek geometers, I call every isosceles triangle whose base angles each measure twice the vertex angle a **Castor Triangle**. That is, the vertex angle measures 36 degrees or $\frac{\pi}{5}$ radians and each base angle measures 72 degrees or $\frac{2\pi}{5}$ radians. This is the triangle that occurs in the construction of a regular decagon as the central triangle determined by two radii of the circle being subdivided into 10 equal parts and a side of the decagon. It is the triangle determined by two diagonals of a regular pentagon emanating from the same vertex and the side opposite that vertex. It is also evident in the ubiquitous 5 pointed star that occurs in the flags of many countries as well as many, many other places, as each of the 5 triangles protruding from the regular pentagon that forms the interior of the star. The ratio of a side length to the base length of a Castor Triangle is the golden ratio $\phi = \frac{\sqrt{5}+1}{2}$ and that is the reason that I call the spiral it determines a Golden Spiral.

The construction of this spiral proceeds as follows. (We use $\doteq$ as an abbreviation of 'defined as'.) Let $\mathcal{C}_0 \doteq [T_0, T_1, T_2]$ denote the Castor Triangle with vertex T_0 and base $[T_1, T_2]$. We bisect the base angle with vertex T_1 and call T_3, the intersection of this bisector with the opposite side $[T_2, T_0]$. Then, because of the angle relationship $\mathcal{C}_1 \doteq [T_1, T_2, T_3]$ is another Castor Triangle. The other triangle thus formed, $\mathcal{P}_1 \doteq [T_3, T_0, T_1]$ is also isosceles. The vertex angle in $\mathcal{P}_1$ measures $\frac{3\pi}{5}$ radians and each base angle measures $\frac{\pi}{5}$ radians, the same as the vertex angle of a Castor Triangle. Still following those ancient Greek geometers that adorn my list of mathematical heroes, I call all isosceles triangles with a vertex angle measuring three times each base angle, a **Pollux Triangle**. In a Pollux Triangle the ratio of the base length to the side length is ϕ. It is the triangle formed by adjacent sides of two adjacent protruding Castor Triangles of a 5 pointed star and the line joining the two vertices. A pair of triangles, one a Castor and the other a Pollux situated so that, like $\mathcal{C}_1, \mathcal{P}_1$, they share a common side and together make a larger Castor Triangle, I call **Gemini Twins**. In this context, the ancient Greek geometers would say that the Pollux Triangle acts as a **gnomon** to transform its Castor Twin into the Mother Castor.

We can now draw the first arc of our golden spiral, from T_0 to T_1 using the vertex T_3 of $\mathcal{P}_1$ as the center and its side length as the radius.

The whole process is now repeated beginning with $\mathcal{C}_1$ as the mother Castor Triangle. The angle with vertex T_2 is bisected. The endpoint of this bisector on $[T_3, T_1]$, we call T_4. Then $\mathcal{C}_2 \doteq [T_2, T_3, T_4]$ is the new Castor Triangle and its twin is $\mathcal{P}_2 \doteq [T_4, , T_1, T_2]$ and the arc drawn circumscribing the base of $\mathcal{P}_2$ from center T_4 is the second arc of our golden spiral. Note that since the centers of the two arcs are collinear with T_1, the tangents to the two arcs at T_1 coincide. Thus the spiral curve is tangent smooth at the transition point.

The nth step in this iterative process begins with the Castor Triangle $\mathcal{C}_n \doteq [T_n, T_{n+1}, T_{n+2}]$. The base angle at T_{n+1} is bisected thus creating a pair of Gemini Twins $\mathcal{C}_{n+1}, \mathcal{P}_{n+1}$. A tangent smooth arc of the associated Golden Spiral is then drawn circumscribing the base of $\mathcal{P}_{n+1}$ using its vertex as the center.

From a transformational viewpoint we map $\mathcal{C}_n$ into $\mathcal{C}_{n+1}$ by rotating the first Castor through $\frac{3\pi}{5}$ radians and simultaneously shrinking its sides by a factor of ϕ. I call this transformation the **Φ-map**. We can think of it as the point transformation that maps each vertex T_n into the 'next' vertex T_{n+1} and describe it by saying that "the Φ-map maps the vertices sequentially".

The nested sequence $\{\mathcal{C}_n\}$ of Castor Triangles arising by the Φ-map converge to a special point E called the **eye** of the spiral. It can be located as the intersection of the medians $[T_{n+2}, L_{n+2}]$ where L_{n+2} is the midpoint of $[T_n, T_{n+1}]$. E has a number of fascinating properties. (For proofs of these properties see [1]).

(1) E divides each median $[T_{n+2}, L_{n+2}]$ in the same ratio: $\frac{[T_{n+2}E]}{[EL_{n+2}]} = \frac{\phi^2}{2}$.

(2) E is the only fixed point of the Φ-map. That is, its location in every Castor Triangle remains the same. In fact the areas of the three triangles into which $\mathcal{C}_n$ is subdivided when E is joined to the three vertices are in the ratio (where, in general, we use $[ABC]$ to denote the area of triangle $[A, B, C]$)

$$[ET_{n+1}T_{n+2}] : [T_nET_{n+2}] : [T_nT_{n+1}E] = 1 : 1 : \phi^2.$$

(3) The Φ-map also maps the midpoint sequence $\{L_{n+2}\}$ sequentially. Indeed each $[L_{n+2}, L_{n+3}, L_{n+4}]$ is a Castor Triangle and thus this sequence determines another golden spiral. This spiral intertwines the original spiral since it has the same eye E.

(4) It is easy to construct the image A_1 under the Φ-map of any point A_0 in the plane $\overline{T_0T_1T_2}$. Simply join A_0 to T_0 and to T_1 and then construct lines parallel to these lines passing through T_1 and T_2 respectively. The intersection of these two lines is A_1. Similarly join A_1 to T_1 and T_2 and then construct lines parallel to these passing through T_2 and T_3 respectively. This will yield A_2, the image of A_1. Of necessity then $[A_0, A_1, A_2]$ will be a Castor Triangle and thus will generate a new golden spiral intertwining the other golden spirals since its eye is also E.

Thus E is simultaneously the eye of an infinite family of golden spirals all of which can be generated from any one of them by the Φ-map.

One other fascinating property of the Castor Sequence is that in each Castor Triangle $\mathcal{C}_n$, the interval $[T_n, T_{n+3}]$ on the side $[T_n, T_{n+2}]$ is divided harmonically by M_{n+1}, the midpoint of $[T_n, T_{n+2}]$ and L_{n+4}, the midpoint of $[T_{n+2}, T_{n+3}]$ (which is a side of $\mathcal{C}_{n+1}$). In other words, $[T_nT_{n+3}]$ is the harmonic mean of $[T_nM_{n+1}]$ and $[T_nL_{n+4}]$. Thus the set $\{T_n, M_{n+1}, T_{n+3}, L_{n+4}\}$ is the geometric counterpart of a major musical chord.

2. The Golden Spiral Inscribed in Golden Rectangles

A golden rectangle is one for which the ratio of the length of the long side to that of the short side is ϕ. This automatically makes the long side length the geometric mean between the semiperimeter and the length of the short side.

Let $\mathcal{R}_1 \doteq [R_1, R_{-2}, R_0, R_2]$ denote a golden rectangle as in Figure 3 with vertical side lengths $[R_1 R_{-2}] = [R_0 R_2] = 1$ and horizontal side lengths $[R_{-2} R_0] = [R_2 R_1] = \phi$.

To construct the golden spiral associated with $\mathcal{R}_1$ we subdivide it by the line segment $[R_3, R_4]$ where R_3 divides $[R_{-2}, R_0]$ and R_4 divides $[R_1, R_2]$ in the golden ratio. This subdivides $\mathcal{R}_1$ into a square $\mathcal{S}_1 \doteq [R_1, R_{-2}, R_3, R_4]$ each of whose sides is of length 1 and a new rectangle $\mathcal{R}_3 \doteq [R_3, R_0, R_2, R_4]$ the ratio of whose sides is $\frac{1}{\phi-1} = \frac{1}{\frac{1}{\phi}} = \phi$ so that $\mathcal{R}_3$ is another golden rectangle. Reversing the process, we can say if to $\mathcal{R}_3$ we adjoin $\mathcal{S}_1$ we end up with $\mathcal{R}_1$ so that $\mathcal{S}_1$ is a gnomon for $\mathcal{R}_3$.

Using R_4 as a center and $[R_4, R_1]$ as the radius we inscribe in $\mathcal{S}_1$, an arc from R_1 to R_3. This is the first arc of our golden spiral.

We now iterate this process using $\mathcal{R}_3$ as the mother rectangle subdividing it into a square $\mathcal{S}_3 \doteq [R_3, R_0, R_5, R_6]$ and a new golden rectangle $\mathcal{R}_5 \doteq [R_5, R_2, R_7, R_8]$. With R_6 as the center we inscribe an arc beginning at R_3 and ending at R_5. Since R_3 is collinear with both centers R_4 and R_6 the two arcs form a tangent smooth curve at R_3.

Each step in the iteration process leads to a subdivision of a golden rectangle $\mathcal{R}_{2d-1}$ by the interval $[R_{2d+1}, R_{2(d+1)}]$ into a square $\mathcal{S}_{2d-1}$ with vertices R_n with $n = 2d-1, 2(d-2), 2(d-1), 2d$ and a new golden rectangle $\mathcal{R}_{2d+1}$ with vertices R_m where $m = 2d+1, 2(d-1), 2d, 2(d+1)$. If d is even the long sides are horizontal, otherwise they are vertical. The arc of the golden spiral associated with this square connects R_{2d-1} to R_{2d+1} and the center is $R_{2(d+1)}$.

The nested sequence of golden rectangles converge to a point E that is the eye of the associated spiral. It may be constructed as the intersection of the diagonals $[R_{-2}, R_2]$ and $[R_0, R_4]$ and it divides each of those diagonals in the ratio ϕ^2. These diagonals are perpendicular.

When R_1, the unused vertex of $\mathcal{R}_1$, is joined to E it intersects the opposite side at R_5. When R_3 the unused vertex of $\mathcal{R}_3$ is joined to E it intersects the opposite side in R_7. These two lines are also perpendicular.

Concerning these four 'spokes' emanating from E we have:

The Golden Harmonic Spokes Theorem

The pencil of four 'spokes' emanating from E (the eye of the golden spiral inscribed in a nested sequence of golden rectangles), consisting of a pair of perpendicular diagonals containing all of the even indexed vertices, and a pair of perpendicular 'non-diagonals' containing all of the odd indexed vertices, contains all of the vertices of all the golden rectangles in the sequence. Each pair bisects the other pair (so that the 8 angles they determine with vertex E are each $\frac{\pi}{4}$ radians and the two pairs together consequently form an harmonic set of lines. Thus they intersect each vertical and each horizontal side in an harmonic set of vertices.

The vertices on each of the four harmonic spokes are marvelously arranged in that any four 'consecutive' (i.e. their indices are of the form $(d, d+4, d+8, d+12)$) vertices on each of them is also an harmonic set of points.

References

[1] Swimmer, A., *On Golden Spirals: The Subtlety of Their Symmetry.* Preprint.

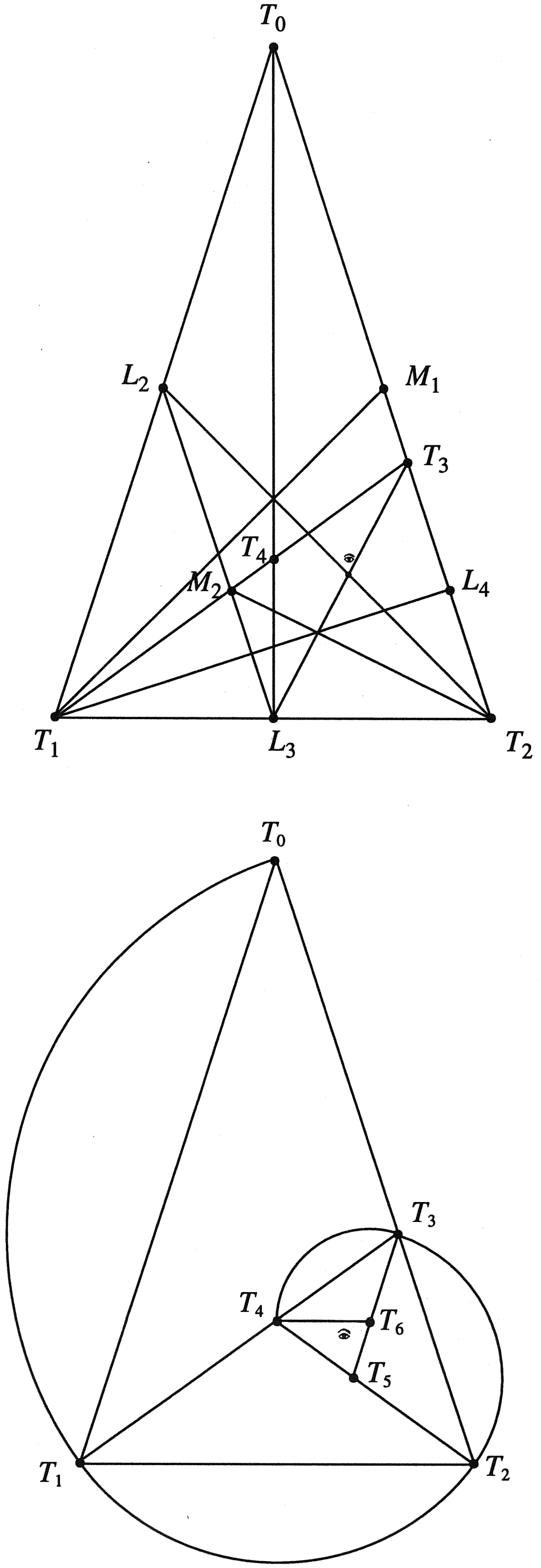

Figure 1. The Gemini Family and its Spiral

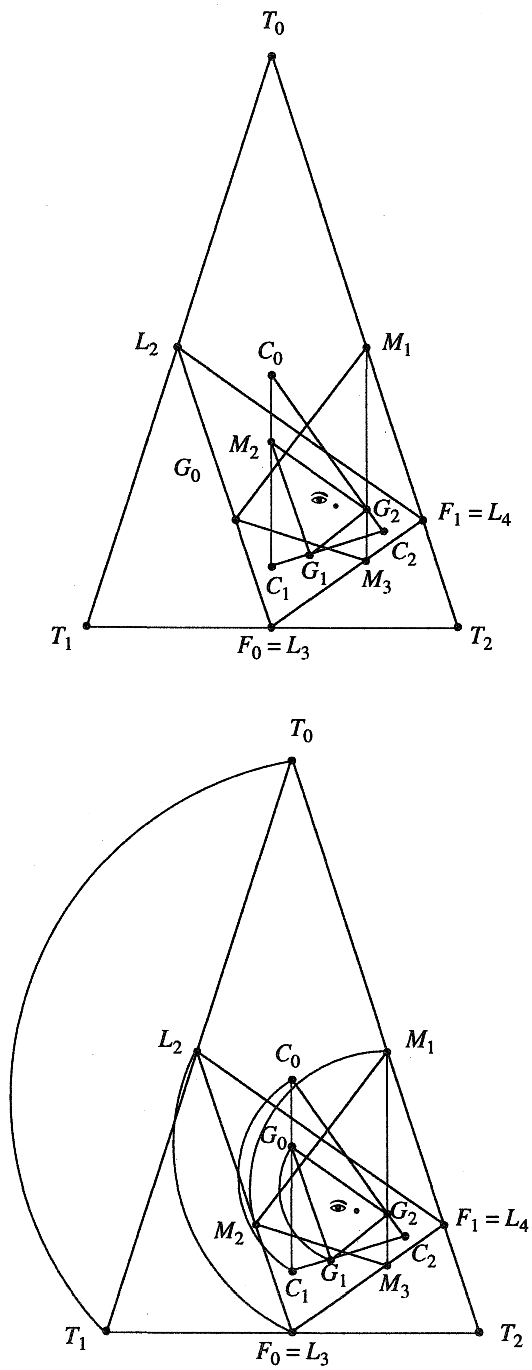

Figure 2. Five Mothers and Their Intertwining Spirals

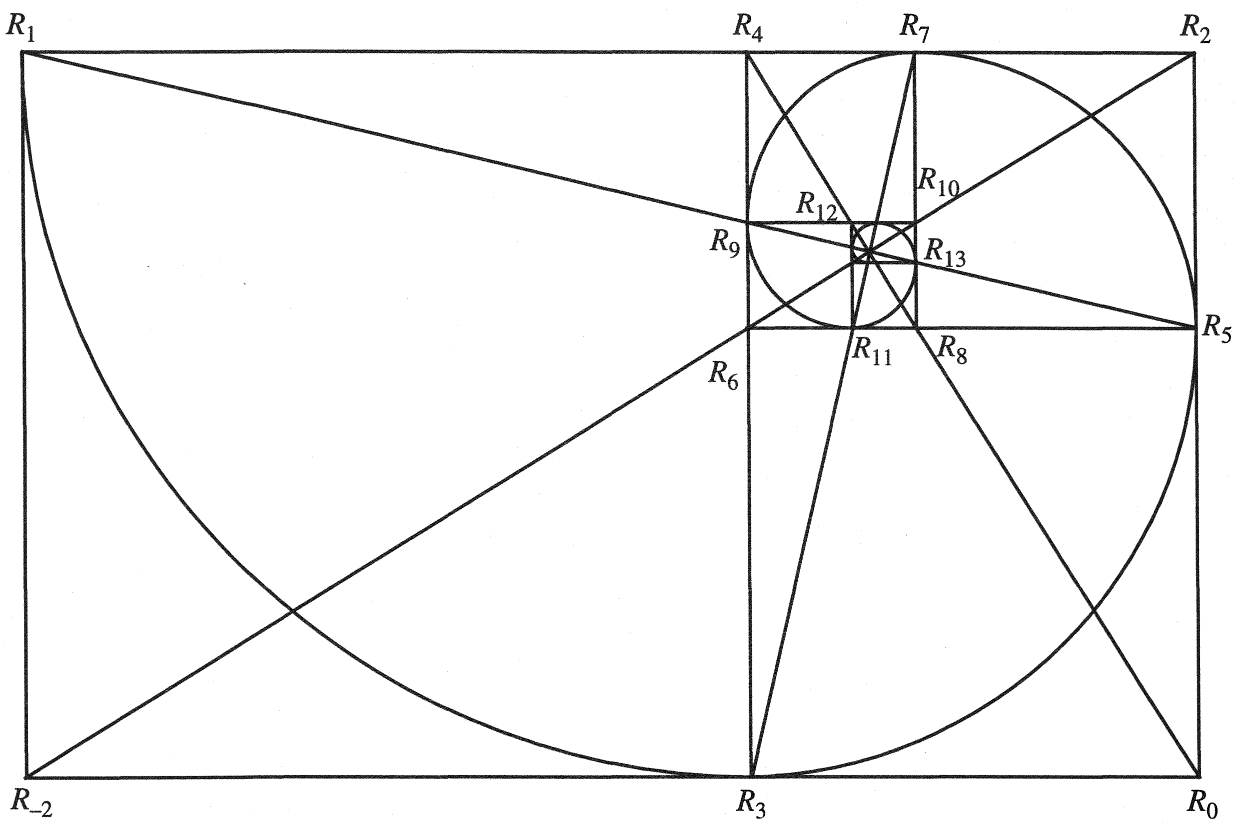

Figure 3. The Pencil of Harmonic Spokes