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Mathematical Connections
in Art, Music, and Science

Art, Math, and Computers: New Ways of Creating Pleasing Shapes

Carlo H. Séquin

Computer Science Division, EECS Department
University of California, Berkeley, CA 94720
E-mail: sequin@cs.berkeley.edu

Abstract

Powerful computers together with sophisticated programs for geometric modeling and graphical rendering allow to create interesting and artistic displays in novel ways. Computers can play a role in the definition of new shapes, e.g., through the use of abstract geometrical generator functions, as well as in the realization of these shapes, e.g., by drawing exact blueprints or by milling parts. On the other hand, "sculptures" described in the computer need not be physically realized but can be enjoyed in their virtual form on a computer display screen or in an ingressive virtual world. Emerging technologies will make it possible to touch these sculptures and to interact with them in cyberspace.

1. Introduction

The urge to create beautiful things that are not exclusively utilitarian may be as old as mankind itself. For tens of thousands of years, people have enhanced their environments with artistic artifacts, decorating bowls and other objects for everyday use with patterns or drawings, or creating figurines or paintings that may have served some ritualistic purposes. Creating pleasing shapes that give the beholder some visual and/or tactile pleasure can take many forms: paintings on cave walls, engravings on clay pots, sculpted bones, or shaped clay figurines. The inspiration for these shapes may come to a large degree from nature; shapes found in nature are stylized, or enhanced, and are placed in interesting constellations. The abstraction of natural shapes can be pushed to a degree that results in purely geometric patterns which may carry some aesthetic appeal without any direct reference to observable shapes in nature; I am thinking of line and dot patterns on Indian vases and baskets, or of the geometrical patterns in Moorish window grills. These patterns derive their visual appeal from their underlying symmetries and from an interplay of repetitiveness and of variation; they often imply several interesting mathematical laws. The growth of our mathematical knowledge during the last few centuries permits us to create geometrical patterns of greater sophistication and in more than just two dimensions.

The introduction of computers enables us to make these patterns more exact or far more rich than would be practical with traditional means. In the last few years, computer graphics capabilities have gained so much power, that high-quality renderings of artistic artifacts can be produced in a fraction of a second. This makes it possible to generate dynamic displays that can be explored in interactive ways, thus creating projections of sculptures in virtual worlds. Unceasingly evolving technology will offer haptic feedback devices that will add a sense of touch to those environments. Virtual sculptures may then appear as real as the stone sculptures carved several thousand years ago. In this paper I focus on a domain where art, math, and computers meet to create pleasing shapes in novel ways, and explore in particular what role the computer can play – today, and in the not too distant future.

2. The Role of the Computer

The process of creating a piece of art has in principle two phases: an inspirational or *design* phase in which the shape of a sculpture is conceived, and an *implementation* phase in which that shape is realized in a particular material. Traditional sculpting often seamlessly blends these two phases; artists draw inspiration from the initial shape of a bone, root, or stone, and then try "to bring out the form that is inherently contained within." During the implementation processes of forming, sculpting, or polishing, the artist "feels" the material and gets stimulation for the further progress of the shaping process. As of today, when computers are involved, the two phases are much more clearly separated. A shape has to be defined and somehow represented in the computer, and then a method has to be found to realize this shape and to make it accessible to a potential audience.

One fairly common way to represent the shape of a sculpture in computer-based form is to cover its surface with hundreds or thousands of points or *vertices* which in turn are interconnected by edges, thereby forming – for instance – a fine mesh of triangles. Each point has to be represented in the computer with three numbers, i.e., the x-, y-, and z- coordinates that define its position in space, and with an identifier that makes it possible to refer to that particular point. In addition, vertices may carry attached coloring and texturing information. Each edge is then defined by reference to the two vertices that it connects; and a triangular facet can be described by reference to three vertices.

One way to make the described shape visible is to send the above data to a *rendering* program. Such a program assumes a certain camera position in space, and from that position projects all visible triangular facets onto the computer display screen, properly taking into account the local intensity variations of the illumination, and perhaps even calculating shadows and/or possible reflections on a shiny surface. The resulting visual effect can be quite realistic if enough computational effort is expended.

Alternatively, the computer model can be used to produce exact blue prints or to guide numerically controlled machine tools to create a physical artifact that realizes the designed model. The computer can thus play a role in both phases: in defining a shape as well as in realizing a conceived shape for enjoyment by others. In the following we will elaborate on the options available in both phases and look at possibilities that will become available in the near future.

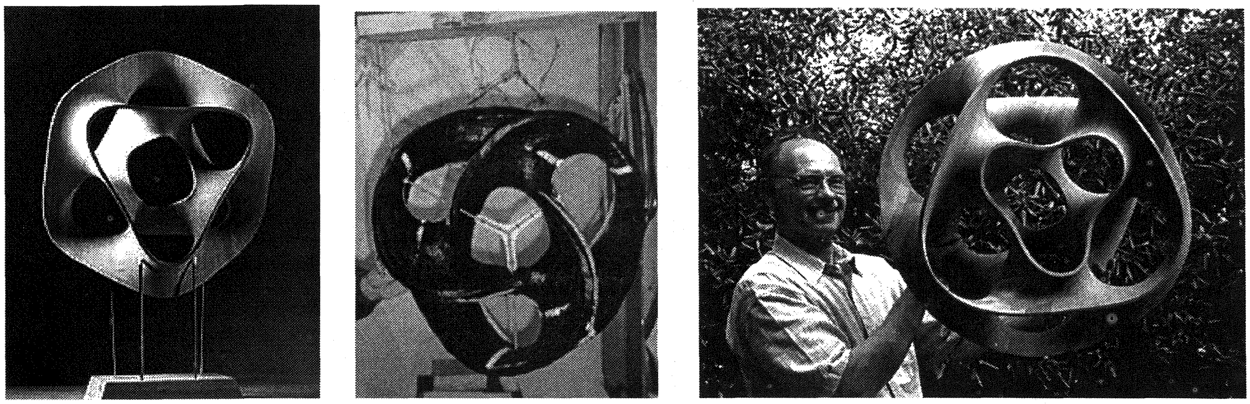
3. Creating a Geometric Computer Model

Typing thousands of vertex coordinates into a computer is a very tedious and error-prone process. It is much easier to start with a suitable initial shape and to make incremental changes with a graphical editing program. The starting shapes could be simple geometrical elements such as spheres, cubes, or cylinders, or they could be drawn from a library of previously modeled parts. A large variety of such geometric computer models are now being sold on CD ROM by various companies.

Interesting shapes can also be captured from existing physical models of moderate sizes (about the size of a human body) with machinery costing several tens of thousands of dollars. Typically a moving swath of laser light illuminates the model, while one or more light detectors or solid state cameras catch the reflected light and determine the 3D positions of the targeted points in space by geometrical triangulation. Sophisticated software then interconnects these data points into a mesh and reduces the description into a simplified geometrical model of acceptable complexity [3]. Any such shape or model can serve as a starting point for further virtual sculpting as described below.

With suitable interactive editing software, individual vertices or groups of vertices can be moved into new positions, or the shape as a whole can be stretched, bent, or twisted. Simulated knives or drills can remove parts of the current shape; but "material" can just as easily be added with other virtual operations. We can glue on or pull out new features, or blend or fuse different shapes together. In this way we can quickly create new shapes that may be quite different from any initial shape.

The use of such software is at least two-fold. The computer could be used as a virtual prototyping tool. Since 1995 I have been collaborating with Brent Collins, an artist who creates wood sculptures of assemblies of saddle surfaces that smoothly flow into one another [7, 2] and which are reminiscent of *minimal surfaces* (see below). Over the phone we discussed one of his recent creations (Fig. 1a), and from our discussion emerged a possibility of providing a ring of connected saddles with some additional partial twist to create a more complicated Moebius-like construction. It took Brent several days to create a rough mock-up from sections of plastic piping, embroidery hoops, and beeswax -- such as the one shown in Figure 1b -- to judge whether the idea had artistic merits. It occurred to me, that with a suitable computer graphics environment, one should be able to evaluate in a matter of hours or minutes whether such an idea was worth pursuing. This prompted me to build a dedicated computer program to simulate and study the toroidal saddle rings that Collins had just started to explore. About two years later, Collins was holding the first sculpture in his hands (Fig. 1c) that was designed in detail with this sculpture generator program [15].



(a) (b) (c)

Figure 1: *Wood sculptures by Brent Collins: (a) designed by intuition, with the help of a mock-up (b) made of rings and beeswax, and (c) designed by Séquin's computer program.*

To many artists, the act of creating a sculpture may be as important as the pleasure of having the final piece. There is joy in working with a physical medium: chiseling away on a marble block, bending a band of hot metal, sanding down a piece of wood, or grinding patterns into a metal surface. Computers can give similar pleasures related to the power of interactive creation – although with entirely different interfaces. While one cannot yet *feel* the emerging forms, one can nevertheless change its shape locally by pushing or pulling on its surface or one can smoothly shape and twist the whole embedding space of a sculpture [14]. In addition, one can also globally change the base material of the sculpture or vary its surface texture. Thus the computer together with appropriate geometrical editing software can become a medium for defining shapes, akin to some kind of virtual clay of infinite flexibility, in which arbitrary shapes can be defined -- with no mess, and where mistakes can readily be undone. A truly interactive editing environment can give the user a tremendous sense of empowerment and can make the creative process as enjoyable and as stimulating as the more traditional and more physical ways of sculpting.

4. Shape Definition by Procedural Generation

Alternatively, initial shape definition or subsequent shape modification can be done in a much more abstract way – by algorithmic computer programs. Here the connection with mathematics becomes the strongest. The "artist," rather than affecting the form directly, specifies procedures, rules, or constraints, based on which a shape -- or a whole family of shapes -- are then generated. This approach can be particularly powerful, since tens of thousands of points can easily be calculated in a very short time.

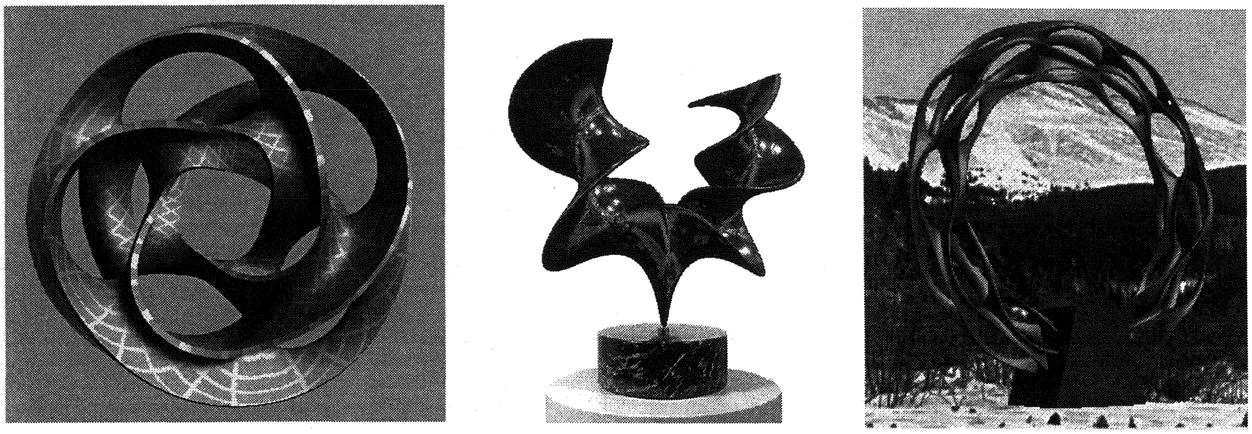

(a) (b) (c)

Figure 2: *Virtual sculptures generated by Séquin's generator program (see text).*

The images in Figure 2 were generated with my generator program [15], specially developed for Brent Collins. The display may look rather geometrical during the initial phases of the design process (Fig. 2a), but with some extra effort can be made to look much more realistic by adding textures and reflections (Fig. 2b), or by painting the sculpture in front of a realistic background (Fig. 2c).

The algorithmic generator approach has other interesting characteristics. The defined shape can be the result of an optimization process that could not be carried out without a computer, i.e., the final shape could be entirely the result of some mathematical concept. In principle such shapes are defined with infinite precision, and it is only a question of how much computational effort one is willing to expend to view these shapes in any desired detail. Moreover, once a generic program is written, many different shapes of a similar nature can be generated by just re-running the program with different parameter settings. As an example in two dimensions, think of the infinite variety of colorful fractal patterns derived from the famous Mandelbrot set [13] by just choosing a different zoom-in window in the display plane. As an example of 3-dimensional shape generation, let's discuss a class of optimally smooth surfaces in which the whole shape is defined entirely by mathematical laws.

5. Shape Definition by Global Optimization

Mother Nature likes to do things efficiently and with the least amount of "cost". For instance, soap films tend to minimize their surface area. When such a film is draped over a warped wire loop, it forms a *minimal surface*: – any small distortion of the naturally assumed shape would be associated with an increase in total surface area. Also, free-floating soap-bubbles are spherical, because that is the container with the smallest surface area for a given volume of air.

Similarly, when bending elastic materials, the natural solutions happen to minimize the total bending energy stored in the deformed material. When a steel rod or a thin wooden slat is pinned down in a few places, they do not form kinks or sharp corners, but rather assume the shape of smooth *spline* curves. Mathematically, the resulting curve is the one that minimizes the integral of the square of curvature along the curve. Such minimum energy curves have long been studied and are used to produce smooth shapes for ship hulls or car bodies [4]. The same principle applies also to two-dimensional thin elastic plates; they minimize the integral of the bending energy taken over its whole surface. This results in smooth shapes known as *minimum energy surfaces*. Computer programs have been developed to calculate the resulting shapes of the surfaces when this type of optimization is applied [1, 8].

Henry Moreton in his Ph.D. dissertation has investigated a new "cost" functional [12] that minimizes the *variation* of curvature, rather than curvature itself. Thus we have to compute integrals over the square of the *change* in curvature when we try to find shapes that minimize this novel *cost function* for which there seems to be no corresponding example in nature. Moreton has developed a computer program [11] to generate such *minimum variation* shapes. A sophisticated optimization loop plays with all degrees of freedom of a given surface to find the shape that minimizes the cost of curvature variation, while keeping the surface within any given external constraints, such as supporting pins and clamps, or specified values for the curvature at some special points.

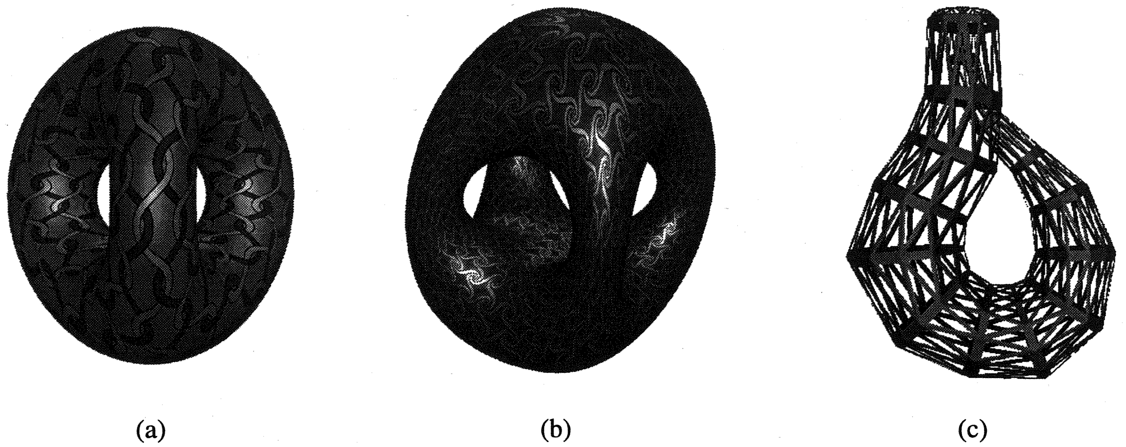

(a) (b) (c)

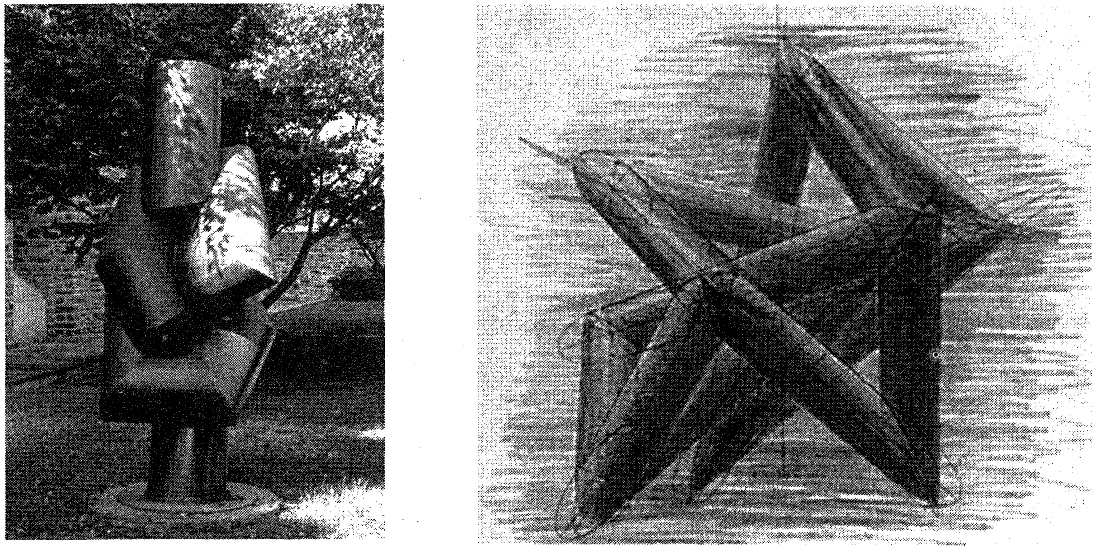
Figure 3: *Procedurally generated shapes: (a, b) Minimum variation surfaces of genus 2 and 3, respectively, and (c) a swept cross section producing the "Skeleton of Klein Bottle."*

The perfect shapes for *minimum variation* curves and surfaces are circles and spheres, respectively. Both of these shapes have the same constant curvature everywhere, and thus their total integrated "cost" is zero. For more complicated shapes with several holes and handles, the surfaces look even smoother and more pleasing than the minimum-energy shapes mentioned above. The shapes displayed in Figures 3a and 3b have been calculated on a Silicon Graphics workstation as true three-dimensional shapes. With the rendering power of these workstations, they can be rotated at interactive speeds, which conveys a good understanding of their true shapes. Since in this article we can present only static renderings, we enhance the comprehension of shape by simulating realistic lighting and by mapping some geometrical textures onto the surfaces.

It should be pointed out, that the generated shape need not correspond to any physically realizable object; disconnected clouds of shapes or self-intersecting surfaces, e.g., Klein bottles, can just as easily be generated (Fig. 3c).

6. Computer Assisted Shape Realization

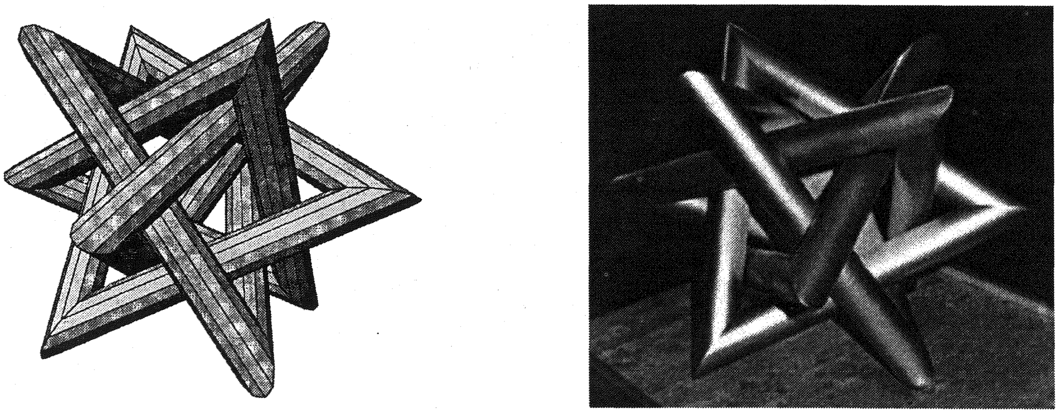
After a computer model of a sculpture has been defined, we still have the problem of getting the shape out of the computer to make it somehow visible to possible observers. This can happen in many different ways. For *constructivist* sculptures, the computer could print out the blueprints for sheet metal parts, or for the exact cutting angles and locations for tubular sections, from which the envisioned shape can then be assembled (Fig. 4a). In 1980 Frank Smullin had developed a simple program running on his Apple II that would plot the outlines of the various tube segments and the elliptical intersections at the joints. By manually applying shading over the geometrically accurate projections of the key features, he could obtain a good visualization of the planned sculpture (Fig. 4b).



(a) (b)

Figure 4: *(a) Tubular Sculpture by Frank Smullin;(b) hand-colored output of his design program.*

Frank Smullin's work [19] inspired me to build the Berkeley Unigrafix System [17,18] and to develop my own sculpture modeling programs [16]. My first program *mkworm* could sweep cross sections along a piecewise linear path in space and produce properly mitred corners that maintain this cross section (Fig. 5a). The program also had facilities to reduce the torsion of the resulting *worm*, by properly subtracting the intrinsic rotation of the Frenet frame, and to add a controlled amount of twist, so that edges of a prismatic tube would properly merge when forming a closed-loop path in space.



(a) (b)

Figure 5: *(a) A design for a sculpture using the Berkeley Unigrafix system, and (b) Séquin's realization.*

Other sculptors prefer to use *subtractive* processes. The actual sculpture itself, or a mold for a cast (Fig. 6a), could be cut from metal, wood, or plastics by numerically controlled milling machines. However, these machines are very expensive and impose strict size limitations. Helaman Ferguson likes to carve stone sculptures of substantial sizes that represent precise mathematical surfaces [5]. In his home he has developed a set-up where his stone drill is suspended by six cables under tension, the length of which is monitored by computer, and from which the coordinates of the tool tip position can be calculated [6].

The computer also calculates the distance from the tool tip to the desired mathematical surface and displays this value on a readout on the drill (Fig. 6b). This makes it possible to start the sculpting process by carving several hundred reference holes to a precise depth.

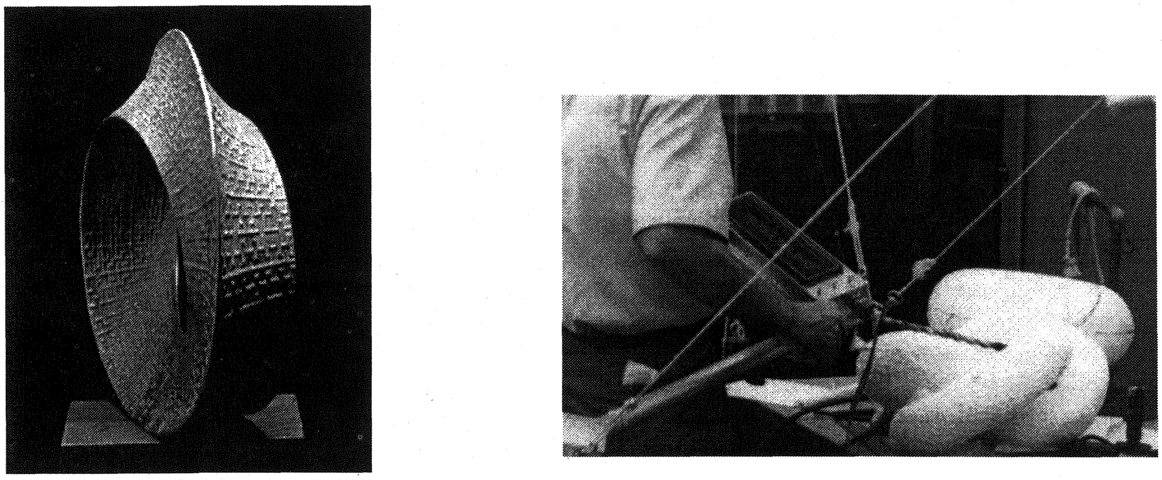

(a) (b)

Figure 6: *Sculpture by H. Ferguson (a) and his set-up for computer controlled sculpting (b).*

Finally, new *solid free-form fabrication* (SFF) technologies [10] are emerging, such as stereo-lithography (SLA) [9], selective laser sintering (SLS), or fused deposition modeling (FDM), in which arbitrary 3-dimensional shapes are built-up layer by layer. In each thin layer, a computer-controlled laser beam polymerizes some liquid plastic resin (SLA), fuses some ceramic powder (SLS), or deposits a thin bead of hot plastics (FDM), thereby adding another contour to the emerging shape. Figure 7 shows two parts that were designed with the Scherk-Collins sculpture generator [15] and which were implemented using different SFF processes. Today these processes are still slow, expensive, and limited in the overall size of the artifacts that can be constructed in a single run.

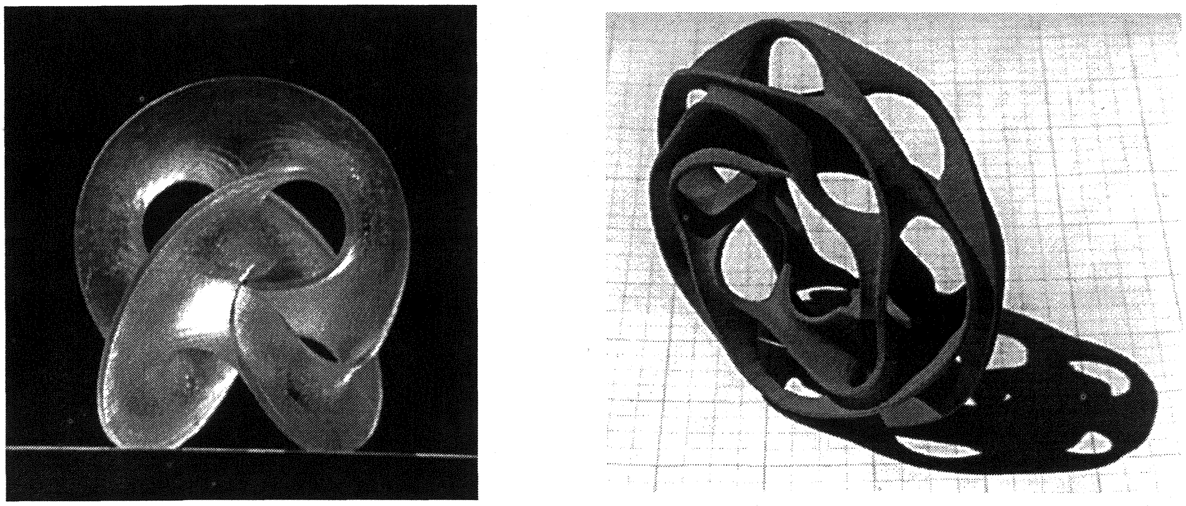

(a) (b)

Figure 7: *SFF: (a) Stereolithography part (made by Metalcast), and (b) part made with Selected Laser Sintering (University of Texas in Austin.*

7. Virtual Shape Realization

Given all the difficulties with physical realization, are there other ways for us to enjoy our possibly quite elaborate computer-generated shapes? We can start by rendering our shapes on a display screen. Using suitable illumination, shading, and shadowing models, we can produce displays that looks like photographs of "real" sculptures. But our computer description is much more than just an "image" of a sculpture; it is a full 3-dimensional model. With enough computer power, we can manipulate this model interactively. We can turn it around faster than we could walk around a large 3D sculpture, and can look at it from angles from which we might never see the real sculpture. We can also view the sculpture under different lighting conditions, we might see mirror reflections of passing clouds in its surface, or we might even change the material from which the sculpture is "constructed."

Moreover, these virtual sculptures can be dynamic; they can change shape as a function of time or in response to user actions such as touching hot buttons or pushing on parts of the sculpture. Figure 8a shows a mechanical assembly that can unfold in one smooth motion from a closed octahedral shape to an open cuboctahedron (truncated cube). For this piece all angles and dimensions have been worked out properly, so that such a mechanism could actually be constructed. But in this virtual machine I did not have to worry about the technical challenges of meeting any strength criteria and of providing a dynamic drive for the whole contraption.

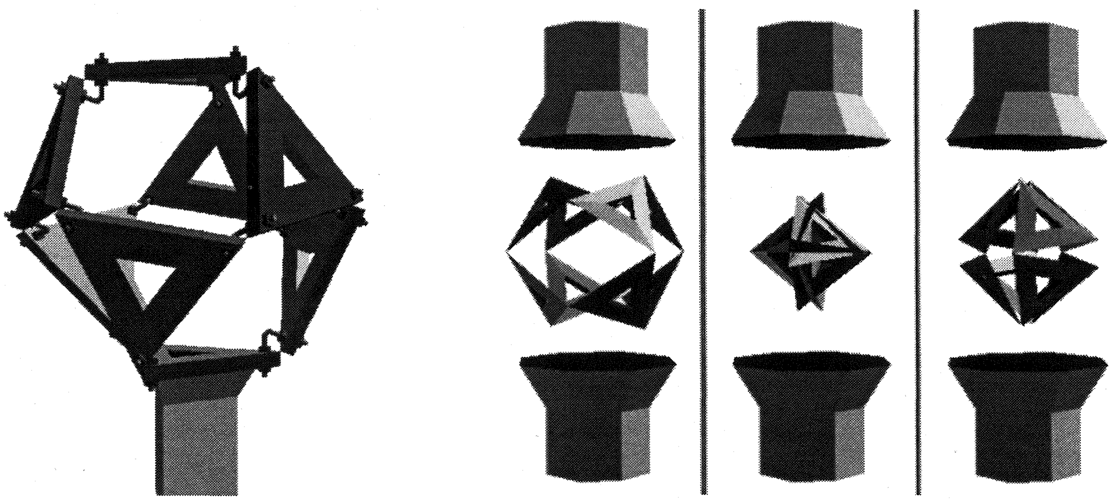

(a) (b)

Figure8: *Realizable (a) and non-realizable (b) virtual dynamic sculptures that change shapes between a closed octahedron and an open cube-octahedron (truncated cube). In (b)* the eight triangular plates intersect each other periodically.

Such virtual sculptures need not even obey our normal physical laws. Parts can float in space, they can change their shapes or sizes, and individual pieces could readily penetrate one another. Figure 8b shows a "magnetically levitated" virtual sculpture in which the plates pass through each other and through the central point periodically so that the whole sculpture turns inside-out.

Such dynamic sculptures could simply move in a periodic manner as a function of time. Alternatively, the user could control the shape transformation with an interactive slider. Similarly, the observer could control a morphing operation that changes one object into another one, as seen in some eye-catching graphics in recent TV ads. More sliders could control more complicated articulated assemblies such as a whole dancing robot with many different joints.

8. The Future of Virtual Reality

The technology for immersive virtual reality environments is under rapid development. Better and less expensive head-mounted displays offering stereo images and "quadraphonic" position-dependent sound, as well as improved input devices and position sensors are emerging every year. Once the problems with resolution, speed, and tracking lag have been overcome, the observers will truly have a sense of being immersed in a synthetically created space. Sensations will not just be limited to video and audio channels; gloves with pressure transducers will make you feel objects as you "touch" them. Excursions to a virtual museum could then become a deceptively real experience. In the long run, visitors will even be able to touch the virtual 3D sculptures and slide their hands along their surfaces; they might even be able to produce a realistic "ring" when they strike a virtual "metal" sculpture.

The most fascinating aspect is that such a virtual environment is not restricted to simulating reality. Invoked reactions could be exaggerated – a small push could result in an unexpectedly large effect, or the reaction could be totally unreal. A virtual sculpture could interact with the observer; it could respond to touch or to the human voice, responding with a change in shape or color, or by producing interesting sounds, music, or voice output. Sculptures could display "feelings;" they could be given individual "personalities," thus reacting differently to the approaches of the observers. The possibilities are boundless and mind-boggling.

9. Conclusions

The combination of art, math, computers, and of other emerging electronic media will offer new possibilities to providing intriguing shapes, sequences of motion, or interactive behavior. These geometrically conceived artifacts may or may not satisfy some yearning for beauty, but, at the very least, they will stimulate our curiosity and cater to our insatiable appetite for novelty.

Procedurally generated computer sculptures – in principle – can be far more complicated than what man could create by hand from physical matter, and their surfaces could be detailed to a degree that would be difficult to achieve with material brushes. The sculptures could be small enough to be "held" in your hand, or they could be as large as a house – and still you can enjoy them right in your living room. Equipped with a head-mounted display and the proper virtual reality set-up, you will be able to fly along the surface of gigantic twisted structure, challenging any real roller-coaster ride for the thrill it produces. The sculptures can also be far more interactive than anything mechanical could be; they may not just consist of articulated rigid parts, but of shapes that morph in intricate preprogrammed ways and which emit coordinated light and sound while they are doing so.

Technology still has a few years to go before we will see flawless and convincing virtual environments at affordable prices. The editing and sculpting tools outlined earlier do not yet exist in the public domain, and significant development is required to produce easy-to-use and intuitive interfaces that will make it as natural to play with these virtual shapes as it is to knead a lump of clay. The problem of creating sculptures in such virtual environments and the potential for new art forms is still poorly understood. We have just started to scratch the surface of a whole new artistic universe.

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